6.003 Signal Processing

Week 4, Lecture A:
Continuous Time Fourier Transform
From Periodic to Aperiodic

- Previously, we have focused on Fourier representations of periodic signals: e.g., sounds, waves, music, ...

- However, most real-world signals are not periodic.

Today: generalizing Fourier representations to include aperiodic signals -> Fourier Transform
Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?

Strategy: make a periodic version of $x(t)$ by summing shifted copies:

$$x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT)$$

Since $x_p(t)$ is periodic, it has a Fourier series (which depends on $T$)

Find Fourier series coefficients $X_p[k]$ and take the limit of $X_p[k]$ as $T \to \infty$

As $T \to \infty$, $x_p(t) \to x(t)$ and Fourier series will approach Fourier transform.
**Fourier Representations of Aperiodic Signals**

\[ x_p(t) = \sum_{m=-\infty}^{\infty} x(t - mT) \]

Calculate the Fourier series coefficients \( X_p[k] \):

\[
X_p[k] = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{-j \frac{2\pi}{T} k t} \, dt = \frac{1}{T} \int_{-S}^{S} 1 \cdot e^{-j \frac{2\pi}{T} k t} \, dt = \frac{1}{T} \int_{-S}^{S} \frac{e^{-j \frac{2\pi}{T} k t}}{(-j 2\pi k / T)} \bigg|_{-S}^{S} = \frac{2\sin\left(\frac{2\pi k}{T} S\right)}{T\left(\frac{2\pi k}{T}\right)}
\]

Plot the resulting Fourier coefficients when \( S=1 \) and \( T=8 \)

What happens if you double the period \( T \)?

There are twice as many samples per period of the sine function

The red samples are at new intermediate frequencies
Fourier Representations of Aperiodic Signals

\[ X_p[k] = \frac{2\sin\left(\frac{2\pi k}{T} S\right)}{T\left(\frac{2\pi k}{T}\right)} \]

Let \( \omega = \frac{2\pi k}{T} \). Define a new function \( X(\omega) = T \cdot X_p[k] = 2 \frac{\sin(\omega S)}{\omega} \)

If we consider \( \omega \) and \( X(\omega) = 2 \frac{\sin(\omega S)}{\omega} \) to be continuous, \( TX_p[k] \) represents a sampled version of the function \( X(\omega) \).

- \( S=1 \) and \( T=8 \):
  \[ TX_p[k] = x\left(\omega = \frac{2\pi k}{T}\right) \]

- \( S=1 \) and \( T=16 \):
  \[ \omega = \frac{2\pi k}{T} \]

- \( S=1 \) and \( T=32 \):
  \[ \omega = \frac{2\pi k}{T} \]

As \( T \) increases, the resolution in \( \omega \) increases.
Fourier Representations of Aperiodic Signals

We can reconstruct $x(t)$ from $X(\omega)$ using Riemann sums (approximating an integral by a finite sum).

$$x_p(t) = \sum_{k=-\infty}^{\infty} X_p[k] e^{j\frac{2\pi}{T}kt} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} TX_p[k] e^{j\frac{2\pi}{T}kt} \left( \frac{2\pi}{T} \right)$$

$$x(t) = \lim_{T \to \infty} x_p(t) = \lim_{T \to \infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} TX_p[k] e^{j\frac{2\pi}{T}kt} \left( \frac{2\pi}{T} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$TX_p[k] = X(\omega)$

As $T \to \infty$,
- $k\omega_0 = \frac{2\pi k}{T}$ becomes a continuum, $\frac{2\pi k}{T} \to \omega$.
- The sum takes the from of an integral, $\omega_0 = \frac{2\pi}{T} \to d\omega$.
- We obtain a spectrum of coefficients: $X(\omega)$. 
Fourier Transform

\[ x(t) = \lim_{T \to \infty} x_p(t) = \lim_{T \to \infty} \frac{1}{2\pi} \sum_k TX_p[k] e^{j \frac{2\pi}{T} k t} \left( \frac{2\pi}{T} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \]

Since \( X(\omega) = T \cdot X_p[k] \)

\[ X_p[k] = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{-j \frac{2\pi}{T} k t} dt \]

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt \]
Continuous-Time Fourier Representations

Fourier series and transforms are similar: both represent signals by their frequency content.

**Continuous-Time Fourier Transform**

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega \\
X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt
\]

**Synthesis equation**

**Analysis equation**

**Continuous-Time Fourier Series**

\[
x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}} \quad \text{Synthesis equation}
\]

\[
X[k] = \frac{1}{T} \int_{T} x(t) e^{-j\frac{2\pi kt}{T}} \, dt \quad \text{Analysis equation}
\]

\[\omega_0 = \frac{2\pi}{T}\]
Continuous-Time Fourier Representations

Periodic signals can be synthesized from a discrete set of harmonics. Aperiodic signals generally require all possible frequencies.

**Continuous-Time Fourier Transform**

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega
\]

\[
X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt
\]

**Continuous-Time Fourier Series**

\[
x(t) = x(t + T) = \sum_{k = -\infty}^{\infty} X[k] e^{j \frac{2\pi kt}{T}}
\]

\[
X[k] = \frac{1}{T} \int_{T} x(t) e^{-j \frac{2\pi kt}{T}} \, dt
\]

\[
\omega_0 = \frac{2\pi}{T}
\]
Continuous-Time Fourier Representations

All of the information in a periodic signal is contained in one period. The information in an aperiodic signal is spread across all time.

**Continuous-Time Fourier Transform**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega$$ \hspace{1cm} \text{Synthesis equation}

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$ \hspace{1cm} \text{Analysis equation}

**Continuous-Time Fourier Series**

$$x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}}$$ \hspace{1cm} \text{Synthesis equation}

$$\omega_0 = \frac{2\pi}{T}$$

$$X[k] = \frac{1}{T} \int_{T} x(t) e^{-j\frac{2\pi k t}{T}} \, dt$$ \hspace{1cm} \text{Analysis equation}
Continuous-Time Fourier Representations

Harmonic frequencies $k \omega_0$ are samples of continuous frequency $\omega$

**Continuous-Time Fourier Transform**

- **Synthesis equation**
  \[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega \]

- **Analysis equation**
  \[ X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt \]

**Continuous-Time Fourier Series**

- **Synthesis equation**
  \[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j \frac{2\pi k t}{T}} \]

- **Analysis equation**
  \[ X[k] = \frac{1}{T} \int_{T} x(t) e^{-j \frac{2\pi k t}{T}} \, dt \]

\[ \omega_0 = \frac{2\pi}{T} \]
Fourier Transform of a Rectangular Pulse

\[ x(t) = \begin{cases} 
1 & -1 < t < 1 \\
0 & \text{otherwise}
\end{cases} \]

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt = \int_{-1}^{1} e^{-j\omega t} \, dt = \frac{e^{-j\omega} - 1}{-j\omega} \bigg|_{-1}^{1} = 2 \frac{\sin(\omega)}{\omega} \]
Fourier Transform of a Rectangular Pulse

The Fourier transform of a rectangular pulse is \(2 \frac{\sin \omega}{\omega}\).

\[X(\omega)\] contains all frequencies \(\omega\) except non-zero multiples of \(\pi\).

\[
X(\omega = m\pi) = \int_{-1}^{1} e^{-j\omega t} dt = \int_{-1}^{1} e^{-jm\pi t} dt = \begin{cases} 2 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}
\]
By definition, the value of $X(\omega = 0)$ is the integral of $x(t)$ over all time
Fourier Transform of a Rectangular Pulse

By definition, the value of $x(t = 0)$ is the integral of $X(\omega)$ over all frequencies, divided by $2\pi$

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega 0} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) d\omega$$
Check yourself!

The signal $x_2(t)$ and its Fourier transform $X_2(\omega)$ are shown below.

Which of the following is true?

1. $b = 2$ and $\omega_0 = \pi/2$
2. $b = 2$ and $\omega_0 = 2\pi$
3. $b = 4$ and $\omega_0 = \pi/2$
4. $b = 4$ and $\omega_0 = 2\pi$
5. none of the above

$$X_2(\omega) = \int_{-\infty}^{\infty} x_2(t) \cdot e^{-j\omega t} \, dt = \int_{-2}^{2} 1 \cdot e^{-j\omega t} \, dt = \frac{e^{-j\omega t}}{-j\omega} \bigg|_{-2}^{2} = 2 \frac{\sin(2\omega)}{\omega} = \frac{4\sin(2\omega)}{2\omega}$$
Stretching In Time

How would $X(\omega)$ scale if time were stretched?

Stretching in time compresses in frequency.
Compressing Time to the Limit

Alternatively, compress time while keeping area = 1:

\[
x(t) = \begin{cases} 
1 & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\
0 & \text{otherwise}
\end{cases}
\]

\[
X(\omega) = \frac{\sin \omega/2}{\omega/2}
\]

In the limit, the pulse has zero width but area 1! We represent this limit with the delta function: \(\delta(t)\).
Math With Impulses

Although physically unrealizable, the impulse (a.k.a. Dirac delta) function $\delta(t)$ is useful as a mathematically tractable approximation to a very brief signal.

$\delta(t)$ only has a nonzero value at $t = 0$, but it has finite area: it is most easily described as an integral:

$$\int_{-\infty}^{\infty} \delta(t) dt = \int_{0_-}^{0_+} \delta(t) dt = 1 \quad \int_{-\infty}^{\infty} \delta(t - a) \ dt = \int_{a_-}^{a_+} \delta(t) \ dt = 1$$

Importantly, it has the following property (the “sifting property”):

$$\int_{-\infty}^{\infty} \delta(t - a) f(t) dt = f(a)$$

let $\tau = t - a$, $\int_{-\infty}^{\infty} \delta(\tau) f(\tau + a) d\tau = \int_{0_-}^{0_+} \delta(\tau) f(a) d\tau = f(a) \cdot \int_{0_-}^{0_+} \delta(\tau) d\tau = f(a)$

The Fourier Transform of $\delta(t)$:

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t} dt = \int_{0_-}^{0_+} \delta(t) \cdot e^{-j\omega t} dt = 1$$
Math With Impulses

Find the function whose Fourier transform is a unit impulse.

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) \cdot e^{j\omega t} \, d\omega = \frac{1}{2\pi} \int_{0-}^{0+} \delta(\omega) \cdot e^{j0t} \, d\omega = \frac{1}{2\pi} \]

\[ 1 \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad 2\pi \delta(\omega) \]

Notice the similarity to the previous result:

\[ \delta(t) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad 1 \]

These relations are **duals** of each other:

- A constant in time consists of a single frequency at \( \omega = 0 \).
- An impulse in time contains components at all frequencies.
Duality

The continuous-time Fourier transform and its inverse are symmetric except for the minus sign in the exponential and the factor of $2\pi$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt \quad \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega$$

If $x(t) \overset{CTFT}{\leftrightarrow} X(\omega)$ then $X(t) \overset{CTFT}{\leftrightarrow} 2\pi x(-\omega)$

How do we show this?

FT synthesis

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega$$

Swapping $\omega$ and $t$

$$x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) \cdot e^{j\omega t} \, dt$$

Change sign of $\omega$

$$x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) \cdot e^{-j\omega t} \, dt$$

Multiply by $2\pi$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) \cdot e^{-j\omega t} \, dt$$
Duality

Duality can be used to simplify math for transform pairs.

\[ x(t) \xrightarrow{FT} X(\omega) \]
\[ \omega \to t \quad t \to -\omega \text{ and scale up by } 2\pi \]
\[ X(t) \xrightarrow{FT} 2\pi x(-\omega) \]
Summary

• Continuous Time Fourier Transform: Fourier representation to all CT signals!
  \[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega \]
  \[ X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt \]

  **Synthesis equation**
  **Analysis equation**

• Very useful signals:
  • Rectangular pulse and its Fourier Transform (sinc)
  • Delta function (Unit impulse) and its Fourier Transform

• Duality:
  • Make it easier to find the FT of new signals
    \[ \text{If } x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \text{ then } X(t) \overset{CTFT}{\longleftrightarrow} 2\pi x(-\omega) \]