6.003 Signal Processing

Week 3, Lecture B:
Discrete Time Fourier Series
Brief Review

Continuous Time Fourier Series

**Synthesis:**

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}} \]

**Analysis:**

\[ X[k] = \frac{1}{T} \int_{T} x(t)e^{-j\frac{2\pi kt}{T}} dt \]

Discrete Time Sinusoids

\[ x[n] = A\cos(\Omega n + \Phi) \]

- n is always an integer!
- Aliasing and base-band

Today: Apply the FS ideas to DT signals and introduce the DT Fourier Series
Check yourself!

What is the fundamental (shortest) period of each of the following signals?

1. \( x_1[n] = \cos \frac{\pi n}{12} \)
2. \( x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15} \)
3. \( x_3[n] = \cos n + \cos 2n + \cos 3n \)

\( x[n] = \cos(\Omega n) \):

If \( x[n] \) is periodic with fundamental period \( N \),
\[
x[n] = x[n + N] = \cos(\Omega(n + N)) = \cos(\Omega n + \Omega N) \quad \rightarrow \quad \Omega N = 2\pi
\]

\( N = \frac{2\pi}{\Omega} \)
Check yourself!

What is the fundamental (shortest) period of each of the following signals?

1. \( x_1[n] = \cos \frac{\pi n}{12} \)
   \[ N = 24 \]

2. \( x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15} \)
   \[ N = 120 \]

3. \( x_3[n] = \cos n + \cos 2n + \cos 3n \)
   \[ N = \infty \]
The period $N$ of a periodic DT signal must be an integer. Therefore the fundamental frequency $\Omega_0 = 2\pi / N$ must be an integer submultiple of $2\pi$.

No such constraints on fundamental frequencies in CT. In CT, the fundamental frequency $\omega_0 = 2\pi / T$ can be any real number.

→ This is an intrinsic difference between CT and DT signals.
Number of Harmonics

• In the case of CTFS, there can be infinite number of harmonics,

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}} \]

• For DT signals with period \( N \), as \( \Omega_0 \) is a submultiple of \( 2\pi \), there are (only) \( N \) distinct complex exponentials \( e^{j\Omega_0 kn} \). The rest harmonics alias.

Example of \( N = 8 \):

\[ \Omega_0 = \frac{2\pi}{8} = \frac{\pi}{4} \]

There are only 8 unique harmonics \( (k\Omega_0) \):

\[ \frac{0\pi}{4}, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}, \frac{6\pi}{4}, \frac{7\pi}{4} \]

or

\[ -\frac{3\pi}{4}, -\frac{2\pi}{4}, -\frac{\pi}{4}, \frac{0\pi}{4}, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4} \]

...
Finitely-many Unique Harmonics

There are $N$ distinct complex exponentials with period $N$.

If a DT signal is periodic with period $N$, then its Fourier series contains just $N$ terms.
Discrete Time Fourier Series

- A DT Fourier Series has just N harmonic frequencies $k\Omega_0$.

\[
    f[n] = c_0 + \sum_{k=0}^{N-1} c_k \cos(k\Omega_0 n) + \sum_{k=0}^{N-1} d_k \sin(k\Omega_0 n)
\]

\[
    f[n] = f[n + N] = \sum_{k=k_0}^{k_0+N-1} F[k] e^{j2\pi\frac{k}{N}n}
\]

where $\Omega_0$ represents the fundamental frequency (radians/sample).

Otherwise, DT Fourier series are similar to CT Fourier series.
Discrete Time Fourier Series

DT Fourier series comprise a weighted sum of just \( N \) harmonics.

\[
x[n] = x[n + N] = \sum_{k=\langle N \rangle} X[k]e^{j\Omega_0 kn}
\]

How to find the weights?

DT Fourier components are also orthogonal:

\[
\sum_{n=n_0}^{n_0+N} e^{j\Omega_0 kn} \cdot e^{-j\Omega_0 mn} = \sum_{n=\langle N \rangle} e^{j\Omega_0 (k-m)n} = \sum_{n=0}^{N-1} (e^{j\Omega_0 (k-m)})^n
\]

\[
= \begin{cases} 
N & \text{if } k = m \\
\frac{1 - (e^{j\Omega_0 (k-m)})^N}{1 - e^{j\Omega_0 (k-m)}} = 0 & \text{if } k \neq m
\end{cases}
\]
Finding DTFS coefficient

Start with DTFS representation:

\[ x[n] = x[n + N] = \sum_{k=\langle N\rangle} X[k] e^{j\Omega_0 kn} \]

Then “sift” out one component \( X[l] \):

\[
\sum_{n=\langle N\rangle} x[n] e^{-j\Omega_0 ln} = \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} X[k] e^{j\Omega_0 kn} e^{-j\Omega_0 ln} = \sum_{k=\langle N\rangle} X[k] \sum_{n=\langle N\rangle} (e^{j\Omega_0 (k-l)} n = N \cdot X[l]
\]

\[ X[k] = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-j\Omega_0 kn} \]
Periodicity with Fourier Series Coefficient $X[k]$ 

Consider a signal $x[\cdot]$ that is periodic in $N$, and consider finding the $(k + N)^{th}$ Fourier Series coefficient:

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega_0 kn}$$
Discrete Time Fourier Series

A periodic DT signal with N samples produces a periodic sequence of N Fourier series coefficients.

\[ X[k] = X[k + N] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\Omega_0 kn} \]

\[ x[n] = x[n + N] = \sum_{k=k_0}^{k_0+N-1} X[k] e^{j\frac{2\pi}{N} kn} \]

DTFS has just N coefficients, whereas CTFS had infinitely many!
Fourier Series Summary

CT and DT Fourier Series are similar, but DT Fourier Series have just $N$ coefficients while CT Fourier Series have an infinite number.

**Continuous-Time Fourier Series**

$$x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}}$$  

**Synthesis equation**

$$\omega_0 = \frac{2\pi}{T}$$

$$X[k] = \frac{1}{T} \int_T^T x(t)e^{-j\frac{2\pi kt}{T}} dt$$  

**Analysis equation**

**Discrete-Time Fourier Series**

$$x[n] = x[n + N] = \sum_{k=k_0}^{k_0+N-1} X[k]e^{j\frac{2\pi kn}{N}}$$  

**Synthesis equation**

$$\Omega_0 = \frac{2\pi}{N}$$

$$X[k] = X[k + N] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\Omega_0 kn}$$  

**Analysis equation**
Check yourself!

- What are the Fourier Series coefficients of the following signal?

\[ x[n] = \begin{cases} 1 & \text{if } n \mod 10 \equiv 0 \\ 0 & \text{otherwise} \end{cases} \]

First, \( x[n] \) is periodic in \( N=10 \)

\[ x[n] = x[n + 10] = \sum_{k=0}^{9} X[k] e^{j \frac{2\pi}{10} kn} \]

\[ X[k] = \frac{1}{10} \sum_{n=0}^{9} x[n] e^{-j \frac{2\pi}{10} kn} = \frac{1}{10} \]

for every \( k \)
Check yourself!

- What are the Fourier Series coefficients of the following signal with a period of $N=10$?

$$ x[n] = 0.5 $$

$$ X[k] = \frac{1}{10} \sum_{n=0}^{9} x[n] e^{-j\frac{2\pi}{10}kn} = \frac{1}{10} \cdot 0.5 \cdot \sum_{n=0}^{9} e^{-j\frac{2\pi}{10}kn} = \begin{cases} 0.5 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} $$
Check yourself!

• What are the Fourier Series coefficients of the following signal?

\[ x[n] = 1 + \cos \left( \frac{2\pi}{5} n \right) \]

First, fundamental frequency \( \Omega_0 = \frac{2\pi}{5} \), signal \( x[n] \) is periodic with \( N = \frac{2\pi}{2\pi/5} = 5 \)

\[ x[n] = 1 + \cos \left( \frac{2\pi}{5} n \right) = 1 + \frac{1}{2} (e^{j\frac{2\pi}{5} n} + e^{-j\frac{2\pi}{5} n}) \]

We have three non-zero \( X[k] \)’s: \( k = 0, k = \pm 1 \)

\[ X[0] = 1 \quad X[1] = \frac{1}{2} \quad X[-1] = \frac{1}{2} \]
Check yourself!

• What are the Fourier Series coefficients of the following signal?

\[ x[n] = 1 + \sin\left(\frac{\pi}{4} n\right) \]

First, fundamental frequency \( \Omega_0 = \frac{\pi}{4} \), signal \( x[n] \) is periodic with \( N = \frac{2\pi}{\pi/4} = 8 \)

\[ x[n] = 1 + \sin\left(\frac{\pi}{4} n\right) = 1 - \frac{j}{2} (e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}) \]

We have three non-zero \( X[k] \)'s: \( k = 0, k = \pm 1 \)

\[ X[0] = 1 \quad X[1] = -\frac{j}{2} \quad X[-1] = \frac{j}{2} \]
Properties of DTFS: Linearity

Consider \( y[n] = A x_1[n] + B x_2[n] \), where \( x_1[n] \) and \( x_2[n] \) are periodic in \( N \). What are the DTFS coefficients \( Y[k] \), in terms of \( X_1[k] \) and \( X_2[k] \)?

First, \( y[n] \) must also be periodic in \( N \)

\[
Y[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} y[n] e^{-j \frac{2\pi}{N} k n} = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} (A x_1[n] + B x_2[n]) e^{-j \frac{2\pi}{N} k n}
\]

\[
= A \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x_1[n] e^{-j \frac{2\pi}{N} k n} + B \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x_2[n] e^{-j \frac{2\pi}{N} k n}
\]

\[
= A X_1[k] + B X_2[k]
\]

If \( y[n] = A x_1[n] + B x_2[n] \), then \( Y[k] = A X_1[k] + B X_2[k] \)
Properties of DTFS: Time flip

• Consider $y[n] = x[-n]$, where $x[n]$ is periodic in $N$. What are the DTFS coefficients $Y[k]$, in terms of $X[k]$?

First, $y[n]$ must also be periodic in $N$

$$Y[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} y[n] e^{-j\frac{2\pi k}{N} n} = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[-n] e^{-j\frac{2\pi k}{N} n}$$

Let $m = -n$

$$Y[k] = \frac{1}{N} \sum_{m=-n_0}^{-n_0+N-1} x[m] e^{-j\frac{2\pi k}{N} (-m)}$$

$$= \frac{1}{N} \sum_{m=-n_0}^{-n_0+N-1} x[m] e^{j\frac{2\pi k}{N} m} = X[-k]$$

If $y[n] = x[-n]$, then $Y[k] = X[-k]$  

Flipping in time flips in frequency.
Properties of DTFS: Time Shift

• Consider $y[n] = x[n - m]$, where $x[n]$ is periodic in $N$, $m$ is an integer. What are the DTFS coefficients $Y[k]$, in terms of $X[k]$?

First, $y[n]$ must also be periodic in $N$

$$Y[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} y[n]e^{-j\frac{2\pi{k}}{N}n} = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n-m]e^{-j\frac{2\pi{k}}{N}n}$$

Let $l = n - m$, then $n = l + m$

$$Y[k] = \frac{1}{N} \sum_{l=n_0-m}^{n_0-m+N-1} x[l]e^{-j\frac{2\pi{k}}{N}(l+m)} = e^{-j\frac{2\pi{k}}{N}m} \cdot \frac{1}{N} \sum_{l=n_0-m}^{n_0-m+N-1} x[l]e^{-j\frac{2\pi{k}}{N}l}$$

$$= e^{-j\frac{2\pi{k}}{N}m} \cdot X[k]$$

If $y[n] = x[n - m]$, then $Y[k] = e^{-j\frac{2\pi{k}m}{N}} X[k]$  

Shifting in time changes phase of Fourier Series Coefficient.
Properties of DTFS: Complex-conjugate Coefficients

If \( x[n] \) is real-valued, \( X[k] = X^*[−k] \).

\[
X[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi k}{N}n}
\]

\[
X[−k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi (−k)}{N}n}
\]

\[
X[−k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{j\frac{2\pi k}{N}n}
\]

\[
X^*[−k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi k}{N}n} = X[k]
\]
Properties of DTFS: Symmetric and Antisymmetric Parts

- A real-valued signal $x[n]$ can be written in terms of the symmetric and antisymmetric parts: $x[n] = x_S[n] + x_A[n]$

\[
x_S[n] = \frac{1}{2} (x[n] + x[-n])
\]
\[
x_A[n] = \frac{1}{2} (x[n] - x[-n])
\]

\[
\frac{1}{2} (X[k] + X[-k]) = \frac{1}{2} (X[k] + X^*[k]) = Re(X[k])
\]

\[
\frac{1}{2} (X[k] - X[-k]) = \frac{1}{2} (X[k] - X^*[k]) = j \cdot Im(X[k])
\]

The real part of $X[k]$ comes from the symmetric part of the signal, the imaginary part of $X[k]$ comes from the antisymmetric part of the signal
Summary

• Discrete-Time Fourier Series:

\[ x[n] = x[n + N] = \sum_{k=\langle N \rangle} X[k]e^{j\frac{2\pi}{N}kn} \]

\[ X[k] = X[k + N] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\Omega_0 kn} \]

discrete in time => periodic in frequency

• Similar properties (as those of CTFS) holds for DTFS: making it easier to find the Fourier coefficients of a new signal