Discrete Cosine Transform

\[ X_C[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) \]

\[ x[n] = X_C[0] + 2 \sum_{k=1}^{N-1} X_C[k] \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) \]
The Discrete Fourier Transform (DFT) implicitly represents the frequencies that are contained in a **periodically extended** version of the input signal, and periodic extension can generate frequencies that are **not present** in the original signal.

Consider the following $8 \times 8$ example.

The brightnesses of left and right edges are different and will generate a sequence of large transitions when periodically extended.
Motivation: 1D

Consider a single row from the previous image. The DFT implicitly extends the signal (here a ramp) periodically.

\[ x[n] = x[n + 8] \]

Although the function is smooth from \( n = 0 \) to 7, the periodic extension contains a series of steps.
Motivation: 1D

We can eliminate the step discontinuities by first replicating one period in reverse order and then extending the result periodically.

\[ y[n] = y[n + 16] \]

The resulting signal is continuous across the edges (however the slope is still discontinuous).
Motivation: 1D

Finally, insert zeros between successive samples.

\[ z[n] = z[n + 32] \]

The resulting signal is real-valued, symmetric about \( n = 0 \), periodic in \( 4N \), and contains only odd numbered samples.

The DFT of this signal is real, symmetric about \( k = 0 \) and anti-periodic. It is completely characterized by \( N \) values: \( Z[0] \) to \( Z[7] \).

This process is captured in the Discrete Cosine Transform.
The Discrete Cosine Transform (DCT) is described by analysis and synthesis equations that are analogous to those of the DFT.

\[
X_C[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cos \left( \frac{\pi k}{N} \left(n + \frac{1}{2}\right) \right) \quad \text{(analysis)}
\]

\[
x[n] = X_C[0] + 2 \sum_{k=1}^{N-1} X_C[k] \cos \left( \frac{\pi k}{N} \left(n + \frac{1}{2}\right) \right) \quad \text{(synthesis)}
\]
Comparison of DFT and DCT Basis Functions

DFT (real and imaginary parts) versus DCT.

\[
\begin{align*}
\text{Re}(e^{j{2\pi k/N}n}) & & \text{Im}(e^{j{2\pi k/N}n}) & & \cos\left(\frac{\pi k}{N}(n + \frac{1}{2})\right) \\
\text{for } k = 0, 1, 2, 3, 4, 5, 6, 7 & & & & \\
\end{align*}
\]
DCT Basis Functions

Much of the utility of Fourier transforms in general and the DFT in particular results from properties of the Fourier basis functions:

- DTFT: \( e^{j\Omega n} \)
- DTFS: \( e^{j\frac{2\pi k}{N} n} \)
- DFT: \( e^{j\frac{2\pi k}{N} n} \)

To better understand the DCT, we need to similarly understand its basis functions.

DCT: \( \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) \)
DCT Basis Functions

The \( k^{th} \) DCT basis function of order \( N \) is given by

\[
\phi_k[n] = \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right).
\]

How many of the following symmetries are true?

- \( \phi_k[n+2N] = \phi_k[n] \)
- \( \phi_k[n+N] = (-1)^k \phi_k[n] \)
- \( \phi_k[n-N] = (-1)^k \phi_k[n] \)
- \( \phi_k[(N-1)-n] = (-1)^k \phi_k[n] \)
DCT Basis Functions

We can use the previous properties to calculate useful facts.

Show that

\[
\sum_{n=0}^{N-1} \phi_k[n] = \sum_{n=0}^{N-1} \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right) = N \delta[k].
\]
Orthogonality

Show that

\[
\sum_{n=0}^{N-1} \phi_k[n] \phi_l[n] = \begin{cases} 
N & \text{if } k = l = 0 \\
N/2 & \text{if } k = l \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

This orthogonality property is the basis of the analysis equation.
Compaction: Gradient

If a signal has predominately low-frequency content, then the higher order coefficients of the DCT tend to decrease faster than the corresponding coefficients of the DFT.

Here are results for a ramp.

Note that the same scales apply for $X_C$ and $X$. 
Compaction: Sinusoids

The same sort of compaction results for sinusoidal signals.

\[
\begin{align*}
\cos(2.1\pi n/N) & \quad \log |X_C[k]| \\
\cos(2.3\pi n/N) & \quad \log |X_C[k]| \\
\cos(2.5\pi n/N) & \quad \log |X_C[k]| \\
\cos(2.7\pi n/N) & \quad \log |X_C[k]| \\
\cos(2.9\pi n/N) & \quad \log |X_C[k]| \\
\end{align*}
\]

Same scales in each panel.