Two-Dimensional DFT

\[ F[k_r, k_c] = \frac{1}{RC} \sum_{r=0}^{R-1} \sum_{c=0}^{C-1} f[r, c] e^{-j \left( \frac{2\pi k_r r}{R} + \frac{2\pi k_c c}{C} \right)} \]

\[ f[r, c] = \sum_{k_r=0}^{R-1} \sum_{k_c=0}^{C-1} F[k_r, k_c] e^{j \left( \frac{2\pi k_r r}{R} + \frac{2\pi k_c c}{C} \right)} \]
Simple Shapes

Find the 2D DFT of the following vertical bar.

Array indices in numpy are \([r, c]\), where \(r\) is row and \(c\) is column. The image is \(32 \times 32\) pixels. The bar is at \(c = 8\).
Simple Shapes

Find the 2D DFT of the following vertical bar.

\[
F[k_r, k_c] = \frac{1}{RC} \sum_{r=0}^{R-1} \sum_{c=0}^{C-1} f[r, c] e^{-j \left( \frac{2\pi k_r}{R} r + \frac{2\pi k_c}{C} c \right)}
\]

\[
= \frac{1}{32^2} \sum_{r=0}^{31} \sum_{c=0}^{31} \delta[c-8] e^{-j \left( \frac{2\pi k_r}{32} r + \frac{2\pi k_c}{32} c \right)}
\]

\[
= \frac{1}{32} \sum_{r=0}^{31} e^{-j \frac{2\pi k_r}{32} r} \frac{1}{32} \sum_{c=0}^{31} \delta[c-8] e^{-j \frac{2\pi k_c}{32} c} = \delta[k_r] \frac{1}{32} e^{-j \frac{2\pi k_c}{32} 8}
\]
Simple Shapes

Find the 2D DFT of the following vertical bar.

\[
f[r, c] = \delta[c-8]
\]

\[
|F[k_r, k_c]| = |\delta[k_r]| \frac{1}{32} e^{-j \frac{2\pi k_c}{32} 8}
\]

Frequency \([k_r, k_c]\) is often plotted with the origin in the center.

How does the \(e^{-j \frac{2\pi k_c}{32} 8}\) term contribute to the right panel?

Could you change \(f[r, c]\) so that \(F[k_r, k_c] = \frac{1}{32} \delta[k_r]\)? (no exponential)

Could you change \(f[r, c]\) so that the horizontal bar in \(F\) is at \(k_r = 8\)?
Simple Shapes

Find the 2D DFT of this image, where bars are at $c=0$ and $c=16$. 
Find the 2D DFT of this image, where bars are at $c=0$ and $c=16$. 

\[
\begin{align*}
\delta[c] & \overset{\text{DFT}}{\leftrightarrow} \frac{1}{32} \delta[k_r] \\
\delta[c] + \delta[c-16] & \overset{\text{DFT}}{\leftrightarrow} \frac{1}{32} \delta[k_r] + \frac{1}{32} e^{-j\frac{2\pi kc}{32}} 16 \delta[k_r] = \frac{1}{32} \left(1 + (-1)^{kc}\right) \delta[k_r] \\
&= \begin{cases} 
\frac{1}{16} & \text{if } k_c \text{ is even and } k_r=0 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]
Simple Shapes

Find the 2D DFT of this image, where bars are at $c=0$ and $c=16$. 

![Diagram showing the 2D DFT of a simple shape with bars at $c=0$ and $c=16$.]
Simple Shapes

Find the 2D DFT of the following image.
Simple Shapes

Find the 2D DFT of the following image.

\[
\begin{align*}
\delta[c] & \quad \overset{\text{DFT}}{\iff} \quad \frac{1}{32} \delta[k_r] \\
\sum_{m=0}^{3} \delta[c-8m] & \quad \overset{\text{DFT}}{\iff} \quad \frac{1}{32} \delta[k_r] \sum_{m=0}^{3} e^{-j \frac{2\pi k_c}{4} 8m} \\
& = \frac{1}{32} \delta[k_r] \sum_{m=0}^{3} e^{-j \frac{2\pi k_c}{4} m} = \frac{1}{8} \delta[k_r] \delta[k_c \text{ mod 4}] 
\end{align*}
\]
Find the 2D DFT of the following image.

$$\sum_{m=0}^{3} \delta[c-8m] \quad \text{DFT} \quad \frac{1}{8} \delta[k_r] \delta[k_c \mod 4]$$
Find the 2D DFT of the following image.
Simple Shapes

Find the 2D DFT of the following image.

\[
\sum_{m=0}^{7} \delta[c - 4m] \quad \text{DFT} \quad \frac{1}{32} \delta[k_r] \quad \sum_{m=0}^{7} \delta[k_c - 4m] \quad \text{DFT} \quad \frac{1}{32} \delta[k_r] \sum_{m=0}^{7} e^{-j \frac{2\pi k_c}{8} 4m}
\]

\[
= \frac{1}{32} \delta[k_r] \sum_{m=0}^{7} e^{-j \frac{2\pi k_c}{8} m} = \frac{1}{4} \delta[k_r] \delta[k_c \text{ mod } 8]
\]
Simple Shapes

Find the 2D DFT of the following image.

What's the relation between the period in space (left) and the period in frequency (right)?
Simple Shapes

Find the 2D DFT of the following image.

What is the effect of the height of the bars?
Simple Shapes

Find the 2D DFT of the following image.

Tall spatial bars $\rightarrow$ short frequency bars, and vice versa.

Height in space times height in frequency $\approx R = 32$. 
Columns

Design a filter to erase the columns from building 10.

What are the distinguishing feature(s) of the columns?
The columns are periodic, with period of about 13.9 pixels.

The columns give rise to vertical stripes in the DFT (shown below).

What is the corresponding spacing of the lines in the DFT?
The columns are periodic, with period of about 13.9 pixels.

Period in time $\times$ period in k-space = width of image.

$$\Delta k_c = \frac{C}{\Delta c} = \frac{256}{13.9} \approx 18.4$$
**Column Filter**

Separate parts of the DFT that correspond to columns (right) from parts that correspond to the rest of the image (left).
Column Filter

Results: without columns (left) versus columns only (right).

What went wrong?
Column Filter Redesign

We classified parts of the DFT near $k_c = 0$ as parts of the columns. However, much of the rest of the image is also near $k_c = 0$.

Reclassify!
Column Filter Redesign

Reclassify parts near $k_c = 0$ as being part of the image, not part of the columns.
Column Filter Redesign

Much better.

Further improvements?

What is causing the artifacts in the sky near the dome?
Column Filter Redesign

Shorten bar height so filter better matches the heights of the columns.
Column Filter Redesign

Now we get a strong effect on the columns and a much weaker effect on the dome.
Compare