Short-Time Fourier Transform and Window Functions
Characterizing Windows

Short-time Fourier transforms are based on the analysis of a sequence of finite-length portions of an input signal.

Goal: Characterize the effect of $N$ on analysis of $x[n]$. 
Characterizing Windows

Each sequence of length $N$ can be thought of as the product of $x[n]$ times a shifted version of a rectangular window $w[n]$ of length $N$:

$$w[n] = \begin{cases} 
1 & \text{if } 0 \leq n < N \\
0 & \text{otherwise}
\end{cases}$$

Find the DTFT of $w[n]$. 

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Find the DTFT of $w[n]$:

$$W(\Omega) = \sum_{n=-\infty}^{\infty} w[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} e^{-j\Omega n} = \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}}$$

$$= e^{-j \frac{\Omega N}{2}} \left( e^{j \frac{\Omega N}{2}} - e^{-j \frac{\Omega N}{2}} \right) = \frac{\sin \frac{\Omega N}{2}}{\sin \frac{\Omega}{2}} e^{-j\Omega \frac{(N-1)}{2}} = W_c(\Omega) e^{-j\Omega \frac{(N-1)}{2}}$$

$$w_c[n]$$

$\Omega$

$-\pi$

0

$\pi$
What is the half-width of the center lobe of $W_c(\Omega)$.

$$W_c(\Omega) = \frac{\sin \frac{\Omega N}{2}}{\sin \frac{\Omega}{2}}$$

Notice that $W_c(\Omega) > 0$ for $-\frac{2\pi}{N} < \Omega < \frac{2\pi}{N}$. Thus the half-width is $\frac{2\pi}{N}$.

The half-width of the central lobe decreases as $N$ increases.
Characterizing Rectangular Windows

Characterize the heights of the sidelobes.

\[ W_c(\Omega) = \frac{\sin \frac{\Omega N}{2}}{\sin \frac{\Omega}{2}} \]

Central lobe has height \( W_c(0) = N \).

Heights of the sidelobes approach \( \frac{1}{\sin(\pi/2)} = 1 \) as \( \Omega \to \pi \).

Ratio of the tallest to shortest lobe is \( \sim N \).
The effect of windowing in time is to convolve in frequency, which blurs the frequency representation.

Two important metrics characterize blur:

- half-width of central lobe $= \frac{2\pi}{N}$ (which decreases with $N$).
- amplitudes of side lobes $> \frac{1}{N}$ (which decreases with $N$).

Increasing $N$ improves both metrics but the improvement is only linear in $N$ while the computation cost (per window) grows faster than $N$ ($N^2$ for direct convolution and $N \log N$ for the FFT).
What if we use a triangular window instead of a rectangular window?

Characterize the blurring properties of a triangular window.
What if we use a triangular window instead of a rectangular window?

Notice that we can think about a triangle window as the convolution of two rectangular windows. Thus the transform of a triangle window is the square of the transform of a square window.

\[ W_c(\Omega) = \frac{\sin \frac{\Omega N}{2}}{\sin \frac{\Omega}{2}} \]

\[ W_t(\Omega) = W_c^2(\Omega) = \frac{\sin^2 \frac{\Omega N}{2}}{\sin^2 \frac{\Omega}{2}} \]
Rectangular and Triangular Windows

Plot rectangular and triangular windows and their transforms.

The triangular window has a narrower central lobe and smaller side-lobes.
Rectangular and Triangular Windows

Use semilog axes for the transforms.

Notice that the triangle window has the same number of sidelobes as the rectangular window, but their amplitudes are much smaller.

However, the triangular window has nearly double the length (in time domain) of the square window!
Rectangular and Triangular Windows

Compare rectangular and triangular windows of equal length.

\[
\log (W_c(\Omega)) \quad \log (W_t(\Omega))
\]

The sidelobes of the triangular window are still much smaller than those of the rectangular window. Why?
Rectangular and Triangular Windows

Compare rectangular and triangular windows of equal length.

The central lobe of the triangular window is twice as wide as that of the rectangular window.
Rectangular and Triangular Windows: Summary

The central lobe of a triangular window is twice as wide as that of the rectangular window with the same length \( N \).

The sidelobes of the triangular window are much smaller than those of a rectangular window (because of Gibb’s phenomenon).

For many purposes (e.g., spectrograms), the smaller sidelobes are more important than the half-width of the central lobe, so higher order windows (triangles, Hann, etc.) are commonly used.