Discrete Fourier Transform (DFT)

Definition and comparison to other Fourier representations.

**DFT:**

\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}
\]

\[
x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}
\]

**DTFS:**

\[
X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi}{N}kn}
\]

\[
x[n] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi}{N}kn}
\]

**DTFT:**

\[
X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}
\]

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega
\]
Analyzing Frequency Content of Arbitrary Signals

Why use a DFT?

Fourier Series: conceptually simple, but limited to periodic signals.

Fourier Transforms: arbitrary signals, but continuous domain \((\omega, \Omega)\).

Discrete Fourier Transform: arbitrary DT signals (finite length)

**Today**: using the DFT to analyze frequency content of a signal.
Create three signals of the following form:

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]
\[ x_2[n] = \cos\left(\frac{8\pi n}{100} - \frac{\pi}{4}\right) \]
\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]
\[ x_4[n] = \cos\left(\frac{9\pi n}{100} - \frac{\pi}{2}\right) \]

Each should have a duration of 1 second and should use a sample frequency of 44,100 Hz.

Compare the DFTs of the first 100 samples of each of these signals.
What is the frequency of this tone if the sample rate is 44,100 Hz?
## Frequency Scales

We can think of the DFT as having spectral resolution of \((2\pi/N)\) radians in DT, which is equivalent to \((f_s/N)\) Hz in CT.

\[
\begin{array}{cccc}
\Omega \text{ [rad/sample]} \\
-\pi & 0 & \pi \\
-N/2 & N/2 \\
-fs/2 & fs/2 \\
\end{array}
\]
What is the frequency of this tone if the sample rate is 44,100 Hz?

\[
f = \frac{\Omega f_s}{2\pi} = \frac{8\pi/100}{2\pi} \times 44,100 \text{ Hz} = 1764 \text{ Hz}
\]

Listen to it.
Write a program to calculate the DFT of an input sequence.
Use that program to calculate $X_1[k]$, which is the DFT of $x_1[n]$. 

$x_1[n] = \cos(8\pi n/100)$
Write a program to calculate the DFT of an input sequence.

Use that program to calculate $X_1[k]$, which is the DFT of $x_1[n]$.

def dft(x):
    N = len(x)
    return [sum([x[n]*e**(-2j*pi*k*n/N) for n in range(N)])/N
             for k in range(N)]

X1 = dft(x[0:100])
Single Sinusoid

Plot the magnitude of $X_1[\cdot]$.

$$X_1[k] = DFT\{x_1[0 : 100]\}$$

Which values of $k$ are non-zero?
Single Sinusoid

Plot the magnitude of $X_1[\cdot]$.

$$X_1[k] = DFT\{x_1[0 : 100]\}$$

Which values of $k$ are non-zero?

$$\Omega = 2\pi k/N = 2\pi k/100 = 8\pi/100; \quad k = 4 \text{ (and } -4)$$
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]
\[ x_2[n] = \cos\left(\frac{8\pi n}{100} - \frac{\pi}{4}\right) \]
Compare Two Signals

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]

\[ x_2[n] = \cos\left(\frac{8\pi n}{100} - \frac{\pi}{4}\right) \]

No difference in magnitudes.
How will plots of DFT magnitudes differ for the following signals?

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]

\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]
Compare Two Signals

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]
\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]

Why are these DFTs so different?
Compare Two Signals

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]
\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]

\[ |X_1[k]| \]
\[ |X_3[k]| \]

\[ \Omega_1 \neq \Omega_3. \] Even more importantly, \( x_3[n] \) is not periodic in \( N = 100! \)
Single Sinusoid

This blurring occurs because the signal is not periodic in the analysis window ($N = 100$).

$$x_3[n] = \cos\left(\frac{9\pi n}{100}\right)$$

What value of $k$ corresponds to $\Omega = \frac{9\pi}{100}$?
Single Sinusoid

This blurring occurs because the signal is not periodic in the analysis window \((N = 100)\).

\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]

What value of \(k\) corresponds to \(\Omega = \frac{9\pi}{100}\)?

\[ \Omega = \frac{9\pi}{100} = \frac{2\pi k}{N} \]

\[ k = 4.5 \]

The signal frequency fell between the analysis frequencies.
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

- $x_3[n] = \cos(9\pi n/100)$
- $x_4[n] = \cos(9\pi n/100 - \pi/2)$
Compare Two Signals

\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]
\[ x_4[n] = \cos\left(\frac{9\pi n}{100} - \frac{\pi}{2}\right) \]

\[ |X_3[k]| \]
\[ |X_4[k]| \]

\[ \Omega_3 = \Omega_4. \] But DC bigger. Higher frequencies smaller. Why?
Analyzing Signals with Multiple Frequencies

What is the minimum window size $N$ needed to resolve $\Omega = \frac{8\pi}{100}$ from $\frac{9\pi}{100}$?

$x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right)$
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 100 \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

\[ \left| X_5[k] \right| \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 100 \text{ zoomed} \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

\[ |X_5[k]| \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 200 \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

\[ |X_5[k]| \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\( N = 200 \) zoomed

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

\[ |X_5[k]| \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 400 \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

\[ |X_5[k]| \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\( N = 400 \) zoomed

\[
x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right)
\]

These frequencies are barely resolved with \( N = 200 \).
Frequency Scales

We can think of the DFT as having spectral resolution of \((2\pi/N)\) radians in DT, which is equivalent to \((f_s/N)\) Hz in CT.

<table>
<thead>
<tr>
<th>(-\pi)</th>
<th>0</th>
<th>(\pi)</th>
<th>(\Omega) [rad/sample]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-N/2)</td>
<td></td>
<td>(N/2)</td>
<td>(k)</td>
</tr>
<tr>
<td>(-f_s/2)</td>
<td></td>
<td>(f_s/2)</td>
<td>(f) [Hz]</td>
</tr>
</tbody>
</table>
Analyzing Signals with Multiple Frequencies

Two frequencies are resolved if they are separated by more than $\frac{2\pi}{N}$.

$\Omega_1 = \frac{8\pi}{100}$ and $\Omega_2 = \frac{9\pi}{100}$ will be resolved if

$$\Delta\Omega = \Omega_2 - \Omega_1 = \frac{9\pi}{100} - \frac{8\pi}{100} = \frac{\pi}{100} > \frac{2\pi}{N}$$

That is, if $N > 200$.

We can think of $\frac{2\pi}{N}$ as the frequency resolution of the DFT.

Notice 8 full cycles of $\Omega_1$ and 9 full cycles of $\Omega_2$ fit in $N = 200$. 