Properties of the Fourier Transform
Continuous-Time Fourier Transform

Synthesis Equation

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \]

Analysis Equation

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \]
Continuous-Time Fourier Transform

Find the Fourier transforms of the following continuous-time signals.

- \( x_1(t) = e^{-t}u(t) \) where \( u(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \)

- \( x_2(t) = e^{-(t-t_0)}u(t - t_0) \)

- \( x_3(t) = \text{Sym}\{e^{-t}u(t)\} \)

- \( x_4(t) = \text{Asym}\{e^{-t}u(t)\} \)

- \( x_5(t) = \frac{d}{dt}\text{Sym}\{e^{-t}u(t)\} \)
Find the Fourier transform of the following signal.

\[ x_2(t) = e^{-(t-t_0)}u(t-t_0) = x_1(t-t_0) \]

\[ X_2(\omega) = e^{-j\omega t_0}X_1(\omega) \]

\[ \angle X_2(\omega) = \angle X_1(\omega) - \omega t_0 \]

Magnitude is unchanged.
Angle offset by straight line through zero (still antisymmetric).
Why does time delay shift phase by angle proportional to frequency?
Continuous-Time Fourier Transform

Why does time delay shift phase by angle proportional to frequency? Think about Fourier components of a signal that are each delayed by same time $t_0$.

The same amount of time corresponds to different amounts of phase.
Continuous-Time Fourier Transform

Find the Fourier transforms of the following continuous-time signals.

• $x_1(t) = e^{-t}u(t)$ where $u(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$

• $x_2(t) = e^{-(t-t_0)}u(t - t_0)$

• $x_3(t) = \text{Sym}\{e^{-t}u(t)\}$

• $x_4(t) = \text{Asym}\{e^{-t}u(t)\}$

• $x_5(t) = \frac{d}{dt}\text{Sym}\{e^{-t}u(t)\}$
Continuous-Time Fourier Transform

Operations in time that map to multiplicative factors in frequency:

\[ x(t) \xrightarrow{\text{ctft}} X(\omega) \]

\[ x(t - t_0) \xrightarrow{\text{ctft}} e^{j\omega t_0} X(\omega) \]

\[ \frac{dx(t)}{dt} \xrightarrow{\text{ctft}} j\omega X(\omega) \]
Discrete-Time Fourier Transform

Synthesis Equation

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \]

Analysis Equation

\[ X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \]
Discrete-Time Fourier Transform

Find the Fourier transforms of the following discrete-time signals.

1. \( x_1[n] = a^n u[n] \) where \( u[n] = \begin{cases} 
1 & \text{if } n \geq 0 \\
0 & \text{otherwise} 
\end{cases} \)

2. \( x_2[n] = a^{(n-n_0)} u[n - n_0] \)

3. \( x_3[n] = \text{Sym}\{a^n u[n]\} \)

4. \( x_4[n] = \text{Asym}\{a^n u[n]\} \)

5. \( x_5[n] = na^n u[n] \)
Discrete-Time Fourier Transform

Find the Fourier transform of $x_2[n] = a^{(n-n_0)}u[n-n_0] = x_1[n-n_0]$.

$$X_2\Omega = e^{-j\Omega n_0} X_1(\Omega)$$

$$|X_2(\Omega)| = |X_1(\Omega)|$$

$$\angle X_2(\Omega) = \angle X_1(\Omega) - \Omega n_0$$

Magnitude is unchanged.
Phase offset by $-\Omega n_0$.

- still antisymmetric?
- still periodic in $2\pi$?
Discrete-Time Fourier Transform

Find the Fourier transforms of the following discrete-time signals.

- \( x_1[n] = a^n u[n] \) where \( u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \)

- \( x_2[n] = a^{(n-n_0)} u[n - n_0] \)

- \( x_3[n] = \text{Sym}\{a^n u[n]\} \)

- \( x_4[n] = \text{Asym}\{a^n u[n]\} \)

- \( x_5[n] = na^n u[n] \)
Find the Fourier transform of $x_6[n]$:

$$x_6[n] = \begin{cases} 
  (\frac{1}{2})^{n/2} & n = 0, 2, 4, 6, 8, \ldots, \infty \\
  0 & \text{otherwise}
\end{cases}$$

Plot the magnitude and angle of $X_6(\Omega)$ versus $\Omega$. 
Inverse Discrete-Time Fourier Transform

Find the signal whose Fourier transform is

\[ X(\Omega) = e^{-j3\Omega} \]