Continuous-Time Fourier Series
(Complex Exponential Form)
Continuous-Time Fourier Series

Complex exponential form.

**Synthesis Equation** (making a signal from components):

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k}{T}t} \]

**Analysis Equation** (finding the components):

\[ X[k] = \frac{1}{T} \int_{T} x(t) e^{-j\frac{2\pi k}{T}t} \, dt \]
Warm Up

Find the Fourier series components $X[k]$ for

$$x(t) = x(t + 2\pi) = \cos(t)$$
Warm Up

Find the Fourier series components \( X[k] \) for

\[
x(t) = x(t + 2\pi) = \cos(t)
\]

We can find \( X[k] \) directly from the synthesis equation:

\[
x(t) = x(t + T) = \sum_{n=-\infty}^{\infty} X[k] e^{j \frac{2\pi k}{T} t}
\]

\[
x(t) = \cos(t) = \frac{1}{2} e^{jt} + \frac{1}{2} e^{-jt}
\]

\[ T = 2\pi \]

\[
X[k] = \begin{cases} 
\frac{1}{2} & k = \pm 1 \\
0 & \text{otherwise}
\end{cases}
\]

Notice that both positive and negative values of \( k \) are needed. (The trig form required just positive values of \( k \).)
Warm Up

Alternatively, we can calculate $X[k]$ from the analysis equation.

$$x(t) = x(t + 2\pi) = \cos(t)$$

$$X[k] = \frac{1}{T} \int_T x(t) e^{-j \frac{2\pi k}{T} t} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t) e^{-j kt} \, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t) (\cos(kt) - j \sin(kt)) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t) \cos(kt) \, dt$$

$$X[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t) \, dt = 0$$

$$X[\pm 1] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2(t) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \cos(2t)\right) \, dt = \frac{1}{2}$$

For $k > 1$:

$$X[\pm k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\cos((k+1)t) + \cos((k-1)t)\right) \, dt = 0$$
Find the Fourier series coefficients $X[k]$ for $x(t)$:
Pulse Train

Find the Fourier series coefficients $X[k]$ for $x(t)$:

$$X[k] = \frac{1}{T} \int_{T} x(t) e^{-j \frac{2\pi k}{T} t} dt$$

$$= \frac{1}{T} \int_{-S}^{S} e^{-j \frac{2\pi k}{T} t} dt = \frac{1}{T} \frac{e^{-j \frac{2\pi k}{T} S} - e^{j \frac{2\pi k}{T} S}}{-j \frac{2\pi k}{T}} = \frac{\sin \left( \frac{2\pi k S}{T} \right)}{\pi k}$$

Notice that $X[k]$ is real-valued:

$$\text{Im} \left( X[k] \right) = 0$$

and $X[k]$ is a symmetric function of $k$:

$$X[-k] = X[k]$$
Properties of Fourier Series

If \( x(t) \) is real-valued, symmetric function of \( t \) then \( X[k] \) is a real-valued, symmetric function of \( k \).

\[
X[k] = \frac{1}{T} \int_{T} x(t) e^{-j \frac{2\pi k}{T} t} dt
\]

Choose symmetric region of integration and expand the exponential.

\[
X[k] = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \left( \cos(2\pi kt/T) - j \sin(2\pi kt/T) \right) dt
\]

If \( x(t) \) is real and symmetric, then the imaginary part integrates to zero.

\[
X[k] = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cos(2\pi kt/T) \, dt
\]

The result is a real-valued and symmetric function of \( k \).
What would happen to Fourier series if you delayed $x(t)$ by $T/2$?

Delay by $T/2$ changes the phase but not the magnitude.

$$x(t) \xleftarrow{\text{ctfs}} X[k]$$

$$x(t - T/2) \xleftarrow{\text{ctfs}} e^{-j\pi k} X[k]$$
Pulse Train

What would happen if you delayed $x(t)$ by $T/4$?

\[ x(t - T/2) \]

Delay by $T/4$ changes the phase but not the magnitude.

\[
x(t) \quad \longleftrightarrow_{\text{ctfs}} \quad X[k] \\
x(t - T/4) \quad \longleftrightarrow_{\text{ctfs}} \quad e^{-j\pi k/2} X[k]
\]
Delay Property of Fourier Series

Delays in time change only the phase of the Fourier series.

\[ X[k] = \frac{1}{T} \int_T x(t)e^{-j\frac{2\pi k}{T}t} \, dt \]

\[ X'[k] = \frac{1}{T} \int_T x(t-t_0)e^{-j\frac{2\pi k}{T}t} \, dt \]

Let \( \tau = t - t_0 \).

\[ X'[k] = \frac{1}{T} \int_T x(\tau)e^{-j\frac{2\pi k}{T}(\tau+t_0)} \, d\tau \]

\[ = e^{-j\frac{2\pi k}{T}t_0} \left( \frac{1}{T} \int_T x(\tau)e^{-j\frac{2\pi k}{T}\tau} \, d\tau \right) = e^{-j\frac{2\pi k}{T}t_0} X[k] \]

\[ x(t) \xleftarrow{\text{ctfs}} X[k] \]

\[ x(t - t_0) \xleftarrow{\text{ctfs}} e^{-j\frac{2\pi k}{T}t_0} X[k] \]
Delay Property of Fourier Series

Complex exponential form simplifies expression of delay property.

<table>
<thead>
<tr>
<th>delay</th>
<th>complex exponential form</th>
<th>trig form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T/2$</td>
<td>mult by $e^{-j\pi k}$</td>
<td>$A'_k = \begin{cases} A_k &amp; k \text{ even} \ -A_k &amp; k \text{ odd} \end{cases}$</td>
</tr>
<tr>
<td>$T/4$</td>
<td>mult by $e^{-j\pi k/2}$</td>
<td>$A'_k = B_k$</td>
</tr>
<tr>
<td>$t_0$</td>
<td>mult by $e^{-j\frac{2\pi k}{T}t_0}$</td>
<td>$B'_k = -A_k$</td>
</tr>
</tbody>
</table>

complicated
Fourier Series Matching

Match the signals (left column) to Fourier series coefficients (right).

$x_1(t)$

$Re \ X_A[k]$  

$Im \ X_A[k]$  

$x_2(t)$

$Re \ X_B[k]$  

$Im \ X_B[k]$  

$x_3(t)$

$Re \ X_C[k]$  

$Im \ X_C[k]$  

$x_4(t)$

$Re \ X_D[k]$  

$Im \ X_D[k]$
Fourier Series Matching

Match the signals (left column) to Fourier series coefficients (right).

\( x_3(t) \) is a real-valued, symmetric function of time. Therefore, \( X_3[k] \) is a real-valued, symmetric function of \( k \).

\[ \rightarrow X_3[k] = X_A[k] \]

\( x_4(t) \) is a real-valued, antisymmetric function of time. Therefore, \( X_4[k] \) is a purely imaginary, antisymmetric function of \( k \).

\[ \rightarrow X_4[k] = X_D[k] \]

\( x_1(t) = x_3(t) + x_4(t) \), therefore \( X_1[k] = X_A[k] + X_D[k] \).

\[ \rightarrow X_1[k] = X_C[k] \]

\( x_2(t) = x_3(t) - x_4(t) \), therefore \( X_2[k] = X_A[k] - X_D[k] \).

\[ \rightarrow X_2[k] = X_B[k] \]