Systems
So far, we have focused our attention on signals and their mathematical representations. However, we are often also interested in manipulating signals. To this end, we introduce the notion of a system (or filter).

Examples:
- audio enhancement: equalization, noise reduction, reverberation, echo cancellation, pitch shift (auto-tune)
- image enhancement: smoothing, edge enhancement, unsharp masking, feature detection
- video enhancement: image stabilization, motion magnification

Example: Running Average
Noisy sensor data can be “smoothed” to reduce the impact of noise on the signal. For example, consider the following data, consisting of a sinusoid corrupted with noise:

Consider the case where this signal is the input to a system described by a “moving average” system, whose output at time $n$ is the average some number of input samples:

$$y[n] = \frac{x[n-1] + x[n] + x[n+1]}{3}$$
**Systems**

We can think of this process as a system:

\[
x[n] \rightarrow \text{3-pt averager} \rightarrow y[n]
\]

The system creates a new signal \(y[n]\) based on its input \(x[n]\). We will characterize this system by the relationship between its input signal and output signals.

Today, we will talk about a few different ways of representing these relationships in both the time domain and the frequency domain.

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**Representations of Signals**

Today, we will consider three different representations of systems.

- **Difference Equation**: represent system by algebraic constraints on samples
- **Convolution**: represent system by its unit sample response
- **Filter**: represent system as amplification/attenuation of frequency components

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**Properties of Systems**

Arbitrary systems are arbitrarily difficult to describe.

Fortunately, many useful systems have two important properties:
- **linearity** (additivity and homogeneity)
- **time invariance**

In 6.003, we will focus on systems that have both of these properties, which we will refer to as **LTI** systems.
Additivity
A system is additive if its response to a sum of inputs is equal to the sum of its responses to each input taken one at a time.

Given
\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]
and
\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]
the system is additive if
\[ x_1[n] + x_2[n] \rightarrow \text{system} \rightarrow y_1[n] + y_2[n] \]
is true for all possible inputs.

Homogeneity
A system is homogeneous if multiplying its input by a constant multiplies its output by the same constant.

Given
\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]
the system is homogeneous if
\[ \alpha x_1[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] \]
is true for all \( \alpha \) and all possible inputs.

Linearity
A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given
\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]
and
\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]
the system is linear if
\[ \alpha x_1[n] + \beta x_2[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] + \beta y_2[n] \]
is true for all \( \alpha \) and \( \beta \) and all possible inputs.

A system is linear if it is both additive and homogeneous.
**Time-Invariance**

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

\[ x[n] \rightarrow \text{system} \rightarrow y[n] \]

the **system is time invariant** if

\[ x[n-n_0] \rightarrow \text{system} \rightarrow y[n-n_0] \]

is true for all \( n_0 \) and for all possible inputs.

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**Check Yourself!**

Consider a system represented by the following difference equation:

\[ y[n] = x[n] + x[n-1] \quad \text{(for all } n) \]

Is this system linear?

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**Check Yourself!**

Consider a system represented by the following difference equation:

\[ y[n] = x[n] + 1 \quad \text{(for all } n) \]

Is this system linear?
Consider a system represented by the following difference equation:
\[ y[n] = x[n] \times x[n-1] \quad \text{(for all } n) \]
Is this system linear?

Consider a system represented by the following difference equation:
\[ y[n] = n \times x[n] \quad \text{(for all } n) \]
Is this system linear?

Consider a system represented by the following difference equation:
\[ y[n] = n \times x[n] \quad \text{(for all } n) \]
Is this system time-invariant?
Representation with Difference Equations

A system is linear and time-invariant if it can be expressed in terms of a linear difference equation with constant coefficients.

General form:
\[ \sum_m c_m y[n-m] = \sum_k d_k x[n-k] \]

**Additivity:** output of sum is sum of outputs
\[ \sum_m (y_1[n-m] + y_2[n-m]) = \sum_k (d_1[n-k] + d_2[n-k]) \]

**Homogeneity:** scaling an input scales its output
\[ \sum_m \alpha y_1[n-m] = \sum_k \alpha d_1[n-k] \]

**Time invariance:** delaying an input delays its output
\[ \sum_m c_m y_1[(n-n_0)-m] = \sum_k d_k x_1[(n-n_0)-k] \]

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Unit Sample Response

If a system is linear and time-invariant, its input-output relation is completely specified by the system’s **unit sample response** \( h[n] \).

The unit sample response \( h[n] \) is the output of the system when the input is the unit sample signal \( \delta[n] \).

The output for more complicated inputs can be computed by summing scaled and shifted versions of the unit sample response.
Superposition

Consider the following signal:

\[ x[n] = \begin{cases} 
1 & \text{if } n = 0 \\
-1 & \text{if } n = 3 \\
-2 & \text{if } n = 4 \\
0 & \text{otherwise} 
\end{cases} \]

This signal can be represented as:

\[ x[n] = \delta[n] - \delta[n-3] - 2\delta[n-4] \]

In general, we can represent a signal as a sum of scaled, shifted deltas:

\[ x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n-m] = \ldots + x[-1] \delta[n+1] + x[0] \delta[n] + x[1] \delta[n-1] + x[2] \delta[n-2] + \ldots \]

If \( h[\cdot] \) is the unit sample response of an LTI system, then the output of that system in response to this arbitrary input \( x[\cdot] \) can be viewed as a sum of weighted and shifted unit sample responses:

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m] = \ldots + x[-1] h[n+1] + x[0] h[n] + x[1] h[n-1] + x[2] h[n-2] + \ldots \]

Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit sample responses.

\[ \delta[n] \xrightarrow{\text{system}} h[n] \]
\[ \delta[n-k] \xrightarrow{\text{system}} h[n-k] \]
\[ x[k] \delta[n-k] \xrightarrow{\text{system}} x[k] h[n-k] \]
\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \xrightarrow{\text{system}} y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]
Superposition by Example

Consider an LTI system described by \( y[n] = 3x[n] - x[n-1] \). Compute the response of this system to an input \( x[n] \) given by:

\[ x[n] = 2\delta[n-1] + 5\delta[n-2] + 3\delta[n-4] \]

Convolution

Response of an LTI system to an arbitrary input.

\[ x[n] \rightarrow \text{LTI} \rightarrow y[n] \]

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n] \]

This operation is called convolution (verb form: convolve).

Convolution

Convolution is represented with an asterisk.

\[ \sum_{m=-\infty}^{\infty} x[m]h[n-m] \equiv (x*h)[n] \]

Convolution operates on signals, not samples.

The symbols \( x \) and \( h \) represent DT signals.

Convolving \( x \) with \( h \) generates a new DT signal \( x*h \).
**DT Convolution: Summary**

Unit sample response $h[n]$ is a complete description of an LTI system.

$\mathbf{x}[n] \rightarrow \mathbf{h}[n] \rightarrow \mathbf{y}[n]$

Given $h[n]$, we can compute the response $y[n]$ to any input $x[n]$ by convolving $x[n]$ and $h[n]$:

$y[n] = (x \ast h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

---

**CT Convolution**

The same sort of reasoning applies to CT signals.

$x(t)$

$x(t) = \lim_{\Delta \to 0} \sum_{k} x(k\Delta)p(t-k\Delta)\Delta$

where $p(t)$

As $\Delta \to 0$, $k\Delta \to \tau$, $\Delta \to d\tau$, and $p(t) \to \delta(t)$:

$x(t) \to \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

---

**Structure of Superposition**

If a system is linear and time-invariant (LTI) then its output is the integral of weighted and shifted unit-impulse responses.

$\mathbf{\delta}(t) \rightarrow \text{system} \rightarrow \mathbf{h}(t)$

$\mathbf{\delta}(t-\tau) \rightarrow \text{system} \rightarrow \mathbf{h}(t-\tau)$

$\mathbf{x}(\tau)\mathbf{\delta}(t-\tau) \rightarrow \text{system} \rightarrow \mathbf{x}(\tau)h(t-\tau)$

$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \text{system} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
**CT Convolution**

Convolution of CT signals is analogous to convolution of DT signals.

**DT:**
\[ y[n] = \{x * h\}[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \]

**CT:**
\[ y(t) = \{x * h\}(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \]

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**Frequency Representation of Convolution**

Let \[ y[n] = \{x * h\}[n] \]. Find \[ Y(\Omega) \].
**Frequency Representation of Convolution**

\[
Y(\Omega) = \sum_{n=-\infty}^{\infty} (h * x)[n] e^{-j\Omega n}
\]

\[
= \sum_{n=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} h[m] x[n-m] \right) e^{-j\Omega n}
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[m] x[n-m] e^{-j\Omega n}
\]

\[
= \sum_{m=-\infty}^{\infty} h[m] \sum_{n=-\infty}^{\infty} x[n-m] e^{-j\Omega n}
\]

\[
= \sum_{m=-\infty}^{\infty} h[m] e^{-j\Omega m} \sum_{l=-\infty}^{\infty} x[l] e^{-j\Omega (l+m)}
\]

\[
= \sum_{m=-\infty}^{\infty} h[m] e^{-j\Omega m} \sum_{l=-\infty}^{\infty} x[l] e^{-j\Omega l} e^{-j\Omega m} = H(\Omega) X(\Omega)
\]

**Filtering**

We can view filtering in both the time and frequency domains:

**Time Domain:**

\[
x[n] \rightarrow h[n] \rightarrow y[n] = (h * x)[n]
\]

**Frequency Domain:**

\[
X(\Omega) \rightarrow H(\Omega) \rightarrow Y(\Omega) = H(\Omega) X(\Omega)
\]

Each frequency component \( X(\Omega) \) is scaled by a factor \( H(\Omega) \), which can be possibly complex.

The system is completely described by the set of scale factors \( H(\cdot) \), which we refer to as the **frequency response** of the system.

**Summary**

**Today:**

- Introduced the notion of a system, including multiple representations:
  - difference equation
  - unit sample response
  - frequency response
- Discussed consequences of focus on LTI systems:
  - time domain: output in response to arbitrary input can be computed by convolving input with unit sample response
  - frequency domain: output in response to arbitrary input can be computed by multiplying input by frequency response

**Next Week:** more on frequency domain interpretation of filtering; designing filters.