6.003: Signal Processing

Signal Processing Systems

- System Abstraction
- Linearity and Time Invariance
- Superposition and Convolution
- Filtering

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Systems

So far, we have focused our attention on signals and their mathematical representations. However, we are often also interested in *manipulating* signals. To this end, we introduce the notion of a system (or filter).

Examples:
- audio enhancement: equalization, noise reduction, reverberation, echo cancellation, pitch shift (auto-tune)
- image enhancement: smoothing, edge enhancement, unsharp masking, feature detection
- video enhancement: image stabilization, motion magnification
Example: Running Average

Noisy sensor data can be “smoothed” to reduce the impact of noise on the signal. For example, consider the following data, consisting of a sinusoid corrupted with noise:

Consider the case where this signal is the input to a system described by a “moving average” system, whose output at time $n$ is the average of some number of input samples:

$$y[n] = \frac{x[n - 1] + x[n] + x[n + 1]}{3}$$
Example: Running Average

\[ y[n] = \frac{x[n - 1] + x[n] + x[n + 1]}{3} \]
We can think of this process as a system:

\[
\begin{align*}
x[n] & \quad \rightarrow \quad 3\text{-pt averager} \\
& \quad \rightarrow \quad y[n]
\end{align*}
\]

The system creates a new signal \( y[n] \) based on its input \( x[n] \). We will characterize this system by the relationship between its input signal and output signals.

Today, we will talk about a few different ways of representing these relationships in both the time domain and the frequency domain.
Representations of Signals

Today, we will consider three different representations of systems.

**Difference Equation**: represent system by algebraic constraints on samples

**Convolution**: represent system by its unit sample response

**Filter**: represent system as amplification/attenuation of frequency components
Properties of Systems

Arbitrary systems are arbitrarily difficult to describe.

Fortunately, many useful systems have two important properties:

- **linearity** (additivity and homogeneity)
- **time invariance**

In 6.003, we will focus on systems that have both of these properties, which we will refer to as LTI systems.
A system is additive if its response to a **sum of inputs** is equal to the **sum of its responses** to each input taken one at a time.

Given

\[
\begin{align*}
    x_1[n] &\rightarrow \text{system} \rightarrow y_1[n] \\
    x_2[n] &\rightarrow \text{system} \rightarrow y_2[n]
\end{align*}
\]

and

the **system is additive** if

\[
\begin{align*}
    x_1[n] + x_2[n] &\rightarrow \text{system} \rightarrow y_1[n] + y_2[n]
\end{align*}
\]

is true for all possible inputs.
Homogeneity

A system is homogeneous if multiplying its input by a constant multiplies its output by the same constant.

Given

\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]

the system is homogeneous if

\[ \alpha x_1[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] \]

is true for all \( \alpha \) and all possible inputs.
Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]

and

\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]

the system is linear if

\[ \alpha x_1[n] + \beta x_2[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] + \beta y_2[n] \]

is true for all \( \alpha \) and \( \beta \) and all possible inputs.

A system is linear if it is both additive and homogeneous.
Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

\[ x[n] \rightarrow \text{system} \rightarrow y[n] \]

the **system is time invariant** if

\[ x[n-n_0] \rightarrow \text{system} \rightarrow y[n-n_0] \]

is true for all \( n_0 \) and for all possible inputs.
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = x[n] + x[n-1] \quad \text{(for all } n \text{)} \]

Is this system linear?
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = x[n] + x[n-1] \quad \text{(for all } n) \]

Is this system linear?

Consider \( x[n] = \alpha x_1[n] + \beta x_2[n] \).

Substituting:

\[
\begin{align*}
    y[n] &= x[n] + x[n-1] \\
    &= (\alpha x_1[n] + \beta x_2[n]) + (\alpha x_1[n - 1] + \beta x_2[n - 1]) \\
    &= (\alpha x_1[n] + \alpha x_1[n - 1]) + (\beta x_2[n] + \beta x_2[n - 1]) \\
    &= \alpha(x_1[n] + x_1[n - 1]) + \beta(x_2[n] + x_2[n - 1]) \\
    \end{align*}
\]

\( \alpha \times \text{output in response to } x_1[\cdot] \quad \beta \times \text{output in response to } x_2[\cdot] \)
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = x[n] + x[n-1] \] (for all \( n \))

Is this system linear?

Consider \( x[n] = \alpha x_1[n] + \beta x_2[n] \).

Substituting:

\[
\begin{align*}
y[n] &= x[n] + x[n-1] \\
&= (\alpha x_1[n] + \beta x_2[n]) + (\alpha x_1[n-1] + \beta x_2[n-1]) \\
&= (\alpha x_1[n] + \alpha x_1[n-1]) + (\beta x_2[n] + \beta x_2[n-1]) \\
&= \alpha (x_1[n] + x_1[n-1]) + \beta (x_2[n] + x_2[n-1]) \\
&= \alpha \times \text{output in response to } x_1[\cdot] + \beta \times \text{output in response to } x_2[\cdot]
\end{align*}
\]

Yes, the system is linear.
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = x[n] + 1 \quad \text{(for all } n) \]

Is this system linear?
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = x[n] + 1 \quad \text{(for all } n) \]

Is this system linear?

Consider \( x[n] = \alpha x_1[n] \).

Substituting:

\[
\begin{align*}
y[n] &= x[n] + 1 \\
&= \alpha x_1[n] + 1 \\
&\neq \alpha (x_1[n] + 1)
\end{align*}
\]
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = x[n] + 1 \quad \text{(for all } n) \]

Is this system linear?

Consider \( x[n] = \alpha x_1[n] \).

Substituting:

\[ y[n] = x[n] + 1 = \alpha x_1[n] + 1 \neq \alpha(x_1[n] + 1) \]

No, the system is not linear.
Check Yourself!

Consider a system represented by the following difference equation:

$$y[n] = x[n] \times x[n-1] \quad \text{(for all } n\text{)}$$

Is this system linear?
Consider a system represented by the following difference equation:

\[ y[n] = x[n] \times x[n-1] \quad \text{(for all } n) \]

Is this system linear?

Consider \( x[n] = \alpha x_1[n] \).

Substituting:

\[
\begin{align*}
y[n] &= x[n] \times x[n-1] \\
     &= (\alpha x_1[n]) \times (\alpha x_1[n-1]) \\
     &= \alpha^2 (x_1[n] \times x_1[n-1]) \\
     &\neq \alpha (x_1[n] \times x_1[n-1])
\end{align*}
\]
**Check Yourself!**

Consider a system represented by the following difference equation:

\[ y[n] = x[n] \times x[n-1] \]  (for all \( n \))

Is this system linear?

Consider \( x[n] = \alpha x_1[n] \).

Substituting:

\[
y[n] = x[n] \times x[n-1] \\
= (\alpha x_1[n]) \times (\alpha x_1[n-1]) \\
= \alpha^2(x_1[n] \times x_1[n-1]) \\
\neq \alpha(x_1[n] \times x_1[n-1])
\]

**No**, the system is not linear.
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = n \times x[n] \]  
(for all \( n \))

Is this system linear?
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = n \times x[n] \quad \text{(for all } n) \]

Is this system linear?

Consider \( x[n] = \alpha x_1[n] + \beta x_2[n] \).

Substituting:

\[
\begin{align*}
  y[n] & = nx[n] \\
  & = n(\alpha x_1[n] + \beta x_2[n]) \\
  & = \alpha(nx_1[n]) + \beta(nx_2[n])
\end{align*}
\]

\( \alpha \times \text{output in response to } x_1[\cdot] \) \hspace{1cm} \( \beta \times \text{output in response to } x_2[\cdot] \)
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = n \times x[n] \quad \text{(for all } n) \]

Is this system linear?

Consider \( x[n] = \alpha x_1[n] + \beta x_2[n] \).

Substituting:

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\begin{align*}
    y[n] &= nx[n] \\
    &= n(\alpha x_1[n] + \beta x_2[n]) \\
    &= \alpha(nx_1[n]) + \beta(nx_2[n])
\end{align*}
\]

\( \alpha \times \text{output in response to } x_1[\cdot] \quad \beta \times \text{output in response to } x_2[\cdot] \)

Yes, the system is linear.
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = n \times x[n] \quad \text{(for all } n) \]

Is this system time-invariant?
Check Yourself!

Consider a system represented by the following difference equation:

\[ y[n] = n \times x[n] \]  (for all \( n \))

Is this system time-invariant?

Consider \( x[n] = x_1[n - n_0] \).

Substituting:

\[ y[n] = nx[n] \]
\[ = nx_1[n - n_0] \]
Consider a system represented by the following difference equation:

\[ y[n] = n \times x[n] \quad \text{(for all } n) \]

Is this system time-invariant?

Consider \( x[n] = x_1[n - n_0] \).

Substituting:

\[
\begin{align*}
y[n] &= nx[n] \\
    &= nx_1[n - n_0]
\end{align*}
\]

This is \textit{not} the same as delaying the original output, which would give \( y[n] = (n - n_0)x_1[n - n_0] \)

\textbf{No}, the system is not time-invariant.
A system is linear and time-invariant if it can be expressed in terms of a linear difference equation with constant coefficients.

**General form:**
\[ \sum_m c_m y[n-m] = \sum_k d_k x[n-k] \]

**Additivity:** output of sum is sum of outputs
\[ \sum_m c_m (y_1[n-m] + y_2[n-m]) = \sum_k d_k (x_1[n-k] + x_2[n-k]) \]

**Homogeneity:** scaling an input scales its output
\[ \sum_m \alpha c_m y_1[n-m] = \sum_k \alpha d_k x_1[n-k] \]

**Time invariance:** delaying an input delays its output
\[ \sum_m c_m y_1[(n-n_0)-m] = \sum_k \alpha d_k x_1[(n-n_0)-k] \]
Representations of Signals

Today, we will consider three different representations of systems.

**Difference Equation**: represent system by algebraic constraints on samples

**Convolution**: represent system by its unit sample response

**Filter**: represent system as amplification/attenuation of frequency components
If a system is linear and time-invariant, its input-output relation is completely specified by the system's unit sample response $h[n]$. The unit sample response $h[n]$ is the output of the system when the input is the unit sample signal $\delta[n]$. The output for more complicated inputs can be computed by summing scaled and shifted versions of the unit sample response.
Superposition

Consider the following signal:

\[
x[n] = \begin{cases} 
1 & \text{if } n = 0 \\
-1 & \text{if } n = 3 \\
-2 & \text{if } n = 4 \\
0 & \text{otherwise}
\end{cases}
\]

This signal can be represented as:

\[x[n] = \delta[n] - \delta[n - 3] - 2\delta[n - 4]\]

In general, we can represent a signal as a sum of scaled, shifted deltas:

\[x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n - m]\]

\[= \ldots + x[-1] \delta[n + 1] + x[0] \delta[n] + x[1] \delta[n - 1] + x[2] \delta[n - 2] + \ldots\]
Superposition

In general, we can represent a signal as a sum of scaled, shifted deltas:

\[ x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n - m] \]

\[ = \ldots + x[-1] \delta[n + 1] + x[0] \delta[n] + x[1] \delta[n - 1] + x[2] \delta[n - 2] + \ldots \]

If \( h[\cdot] \) is the unit sample response of an LTI system, then the output of that system in response to this arbitrary input \( x[\cdot] \) can be viewed as a sum of scaled, shifted unit sample responses:

\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] \]

Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit sample responses.

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \rightarrow \text{system} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]
Consider an LTI system described by \( y[n] = 3x[n] - x[n - 1] \). Compute the response of this system to an input \( x[\cdot] \) given by:

\[
x[n] = 2\delta[n - 1] + 5\delta[n - 2] + 3\delta[n - 4]
\]
Convolution

Response of an LTI system to an arbitrary input.

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \equiv (x \ast h)[n] \]

This operation is called \textit{convolution} (verb form: \textit{convolve}).
Convolution

Convolution is represented with an asterisk.

\[ \sum_{m=-\infty}^{\infty} x[m] h[n - m] \equiv (x * h)[n] \]

Convolution operates on **signals**, not samples.

The symbols \(x\) and \(h\) represent DT signals.

Convolving \(x\) with \(h\) generates a new DT signal \(x * h\).
DT Convolution: Summary

Unit sample response $h[n]$ is a complete description of an LTI system.

![Diagram](image)

Given $h[\cdot]$, we can compute the response $y[\cdot]$ to any input $x[\cdot]$ by convolving $x[\cdot]$ and $h[\cdot]$:

$$y[n] = (x * h)[n] \equiv \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$
The same sort of reasoning applies to CT signals.

\[ x(t) = \lim_{\Delta \to 0} \sum_{k} x(k\Delta)p(t - k\Delta)\Delta \]

where

As \( \Delta \to 0 \), \( k\Delta \to \tau \), \( \Delta \to d\tau \), and \( p(t) \to \delta(t) \):
Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the integral of weighted and shifted unit-impulse responses.

\[ x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau \]

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]
Convolution of CT signals is analogous to convolution of DT signals.

**DT:** \( y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k] \)

**CT:** \( y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \)
Representations of Signals

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Frequency Representation of Convolution

Let \(y[n] = (x \ast h)[n]\). Find \(Y(\Omega)\).
Frequency Representation of Convolution

\[
Y(\Omega) = \sum_{n=-\infty}^{\infty} (h * x)[n]e^{-j\Omega n}
\]

\[
= \sum_{n=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} h[m]x[n - m] \right) e^{-j\Omega n}
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[m]x[n - m]e^{-j\Omega n}
\]

\[
= \sum_{m=-\infty}^{\infty} h[m] \sum_{n=-\infty}^{\infty} x[n - m]e^{-j\Omega n}
\]

\[
= \sum_{m=-\infty}^{\infty} h[m] \sum_{l=-\infty}^{\infty} x[l]e^{-j\Omega (l+m)}
\]

\[
= \sum_{m=-\infty}^{\infty} h[m] \sum_{l=-\infty}^{\infty} x[l]e^{-j\Omega l}e^{-j\Omega m}
\]

\[
= \sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega m} \sum_{l=-\infty}^{\infty} x[l]e^{-j\Omega l} = H(\Omega)X(\Omega)
\]
Filtering

We can view filtering in both the time and frequency domains:

Time Domain:

\[ x[n] \quad h[n] \quad y[n] = (h \ast x)[n] \]

Frequency Domain:

\[ X(\Omega) \quad H(\Omega) \quad Y(\Omega) = H(\Omega)X(\Omega) \]

Each frequency component \( X(\Omega) \) is scaled by a factor \( H(\Omega) \), which can be possibly complex.

The system is completely described by the set of scale factors \( H(\cdot) \), which we refer to as the **frequency response** of the system.
Summary

Today:

- Introduced the notion of a system, including multiple representations:
  - difference equation
  - unit sample response
  - frequency response
- Discussed consequences of focus on LTI systems:
  - time domain: output in response to arbitrary input can be computed by convolving input with unit sample response
  - frequency domain: output in response to arbitrary input can be computed by multiplying input by frequency response

Next Week: more on frequency domain interpretation of filtering; designing filters.