This Week

This coming Friday (20 Sep) is a student holiday. As a result:
- We will still have office hours on Friday 1-3pm, but
- Lab 3 checkoff will be due Sunday evening at 9pm.

The rest of pset 3 releases and is due as normal.

Last Time

Last time:
- Complex number review
- CTFS Complex exponential form
- Properties of CTFS
Check Yourself!

Which of the following is/are true?

• \( \frac{1}{\cos(\theta) + j \sin(\theta)} = \cos(\theta) - j \sin(\theta) \)

• \((\cos(\theta) + j \sin(\theta))^n = \cos(n\theta) + j \sin(n\theta)\)

• \(|2 + 2j + e^{j\pi/4}| = |2 + 2j| + |e^{j\pi/4}|\)

• \(\text{Im} (j^2) > \text{Re} (j^2)\)

• \(\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}(1)\)

Fourier Analysis

Over the last two weeks, we have been introducing the notion of Fourier analysis: thinking of signals as sums of sinusoids (or, equivalently, as sums of complex exponentials).

We choose sinusoids for a number of reasons:
• interesting mathematical properties (such as orthogonality)
• prevalence in nature and biophysics

We generally prefer the complex exponential form in particular because it is mathematically convenient.

CTFS: Complex Exponential Form

Represent a periodic signal as a sum of harmonically-related complex exponentials:

Synthesis:

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T} \]

Analysis:

\[ X[k] = \frac{1}{T} \int_{T} x(t) e^{-j2\pi kt/T} dt \]
Today: **DTFS**

Today, we’ll apply those same ideas to DT signals and introduce the DT Fourier Series.

**Why do we care about DT?**

Before we dive in, let’s first review a little bit about DT sinusoids.

---

**DT Sinusoids**

In general, a DT sinusoid has the form:

\[ x[n] = A \cos(\Omega n + \phi) \]

- \( A \) is referred to as the amplitude
- \( \Omega \) is referred to as the discrete frequency
- \( \phi \) is referred to as a phase offset

Importantly, \( n \) is always an integer!

---

**DT Sinusoids: Aliasing**

Because \( n \) is an integer, we can only faithfully represent frequencies in the **base band**: \( 0 \leq \Omega \leq \pi \).

Frequencies outside that range alias to frequencies in that range.

For example, the following graphs are of cosines with \( \Omega = 0.2\pi, 2.2\pi, \) and \( 4.2\pi \), respectively:

They all produce exactly the same samples!

Today: Fourier analysis on periodic DT signals, including consequences of aliasing.
Discrete-time Fourier Series

DTFS: represent periodic DT signal as a sum of weighted harmonics of a fundamental discrete frequency $\Omega = \frac{2\pi}{N}$:

$$x[n] = x[n + N] = \sum_k X[k]e^{j2\pi kn/N}$$

We can solve for $X[k]$ in a similar fashion to CTFS: multiply by basis function and sum over one period:

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

These are very similar to the CTFS equations, but they differ in important ways!

Finitely-many Unique Harmonics

A crucial observation is that, for a given fundamental frequency $\frac{2\pi}{N}$, there are only $N$ unique harmonics, due to aliasing!

Consider $y[n] = e^{j\Omega n}$, which is periodic in $N$. Given its periodicity in $N$, we can say that:

$$y[n] = e^{j\Omega n} = y[n + N] = e^{j\Omega(n + N)}$$

Thus, $e^{j\Omega N} = 1$, and so $\Omega$ must be one of the $N^{th}$ roots of 1.

Example: $N = 8$:

\[ \text{Re} \quad \text{Im} \]

Finite many Unique Harmonics

Another justification: consider a signal $x[n]$ that is periodic in $N$, and consider finding the $(k + N)^{th}$ FSC:

$$X[k + N] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi(k + N)n}{N}}$$

\[ \text{Re} \quad \text{Im} \]
**Discrete-time Fourier Series**

A periodic DT signal with \( N \) samples produces a periodic sequence of \( N \) Fourier series coefficients.

\[
X[k] = X[k + N] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n]e^{-j2\pi kn/N}
\]

\[
x[n] = x[n + N] = \sum_{k=k_0}^{k_0+N-1} X[k]e^{j2\pi kn/N}
\]

DTFS has just \( N \) coefficients, whereas CTFS had infinitely many!

**Example**

What are the Fourier Series coefficients of the following signal?

\[
x[n] = \begin{cases} 
1 & \text{if } n \mod 10 = 0 \\
0 & \text{otherwise}
\end{cases}
\]
Example
What are the Fourier Series coefficients of the following signal?

\[ x[n] = 1 + \sin\left(\frac{\pi}{4} n\right) \]

Example
What are the Fourier Series coefficients of the following signal?

\[ x[n] = 1 + \cos\left(\frac{\pi}{4} n\right) \]

Example
What are the Fourier Series coefficients of the following signal?

\[ x[n] = \frac{5}{2} - 3 \cos\left(\frac{2\pi}{5} n\right) + \frac{3}{2} \sin\left(\frac{2\pi}{4} n\right) \]
Example
Consider a family of signals $x_m[\cdot]$, all periodic in $N = 30$, where

$$x_m[n] = \cos \left( \frac{2\pi mn}{30} \right)$$

How do $X_1[\cdot], X_2[\cdot], \ldots, X_{14}[\cdot]$ compare?
(imagine plotting them vs $k$)

How do $x_1[\cdot], x_2[\cdot], \ldots, x_{14}[\cdot]$ compare?
(imagine plotting them vs $n$)

Example
Consider a family of signals $x_m[\cdot]$, all periodic in $N = 30$, where

$$x_m[n] = \cos \left( \frac{2\pi mn}{30} \right)$$

How do $X_2[\cdot]$ and $X_{28}[\cdot]$ compare?
(imagine plotting them vs $k$)

How do $x_2[\cdot]$ and $x_{28}[\cdot]$ compare?
(imagine plotting them vs $n$)

Example
Consider a family of signals $x_m[\cdot]$, all periodic in $N = 30$, where

$$x_m[n] = \cos \left( \frac{2\pi mn}{30} \right)$$

What is $x_{15}[\cdot]$?

What is $X_{15}[\cdot]$?
**Properties: Linearity**

Operations on the time-domain representation can often be interpreted as equivalent operations on the FSC.

For example, let \( x[n] = Ax_1[n] + Bx_2[n] \), where \( x_1[n] = x_1[n + N] \) and \( x_2[n] = x_2[n + N] \). Then,

\[
X[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\frac{2\pi kn}{N}} = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} (Ax_1[n] + Bx_2[n])e^{-j\frac{2\pi kn}{N}}
\]

\[
= A\frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x_1[n]e^{-j\frac{2\pi kn}{N}} + B\frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x_2[n]e^{-j\frac{2\pi kn}{N}}
\]

\[
= AX_1[k] + BX_2[k]
\]
Properties: Complex-conjugate Coefficients

If $x[n]$ is real-valued, $X[k] = X^*[−k]$.

$$X[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j \frac{2\pi kn}{N}}$$

$$X[-k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{j \frac{2\pi kn}{N}} = X^*[k]$$

Properties: Symmetric and Antisymmetric Parts

FSC for the symmetric and antisymmetric parts of a real-valued signal, $x[n] = x_s[n] + x_a[n]$.

$$x_s[n] = \frac{1}{2} (x[n] + x[-n]) \quad \text{DTFS} \quad \frac{1}{2} (X[k] + X^*[k]) = \text{Re}(X[k])$$

$$x_a[n] = \frac{1}{2} (x[n] - x[-n]) \quad \text{DTFS} \quad \frac{1}{2} (X[k] - X^*[k]) = j\text{Im}(X[k])$$

Summary

Today, we introduced the DTFS and talked about the consequences of aliasing (discrete in time $\rightarrow$ periodic in frequency).

Recitation 3A: Use DTFS to understand an auditory phenomenon

Lecture 3B: Leveraging properties of DTFS

No Recitation 3B

Next Week: Applying Fourier analysis to aperiodic signals