Transforms

Signals are functions that convey information. Sometimes, though, the straightforward way of representing a signal may not expose important properties of the signal.

It is often useful to have multiple different ways of looking at a signal.

For example, consider the following two representations of a speech signal:
**Transforms**

We call such an alternative view of a signal a **transform**.

In 6.003, we will focus primarily on a family of transforms referred to as **Fourier transforms**, where signals are represented as sums of sinusoids, for example:

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_0 t + d_k \sin k\omega_0 t)$$

PSet 1 contained an example of this idea. Given a bunch of coefficients associated with harmonically-related sinusoids, find the waveform that results from combining them together.

**Today:**

How can we go in the other direction? Under what conditions can \( f(t) \) be expressed in this form? How can we compute \( c_k \) and \( d_k \)?

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**Preliminaries: Terminology**

A **periodic** signal is one that repeats after some amount of time \( T \), such that, for all times \( t \),

$$x(t) = x(t + T)$$

A function that is periodic in \( T \) is also periodic in \( 2T, 3T, 4T, \ldots \). For a given signal, the smallest value \( T \) for which the above holds is called that signal’s **fundamental period**.

The associated frequency, \( \omega_0 = \frac{2\pi}{T} \), is the signal’s **fundamental frequency**.

Frequencies are **harmonically related** if they are integer multiples of some fundamental frequency.

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**Check Yourself**

Estimate the shape of the following two signals:

- \( f_1(t) = \cos(2\pi t) + 0.5 \cos(2\pi t) \)
- \( f_2(t) = \cos(2\pi t) + 0.5 \cos(2\pi t + \pi/4) \)
- \( f_3(t) = \cos(2\pi t) + 0.5 \cos(2\pi t - \pi/4) \)
**Preliminaries: Sinusoids**

Summing two sinusoids with the same frequency yields a (scaled, shifted) sinusoid at that same frequency.

\[ A \cos(\omega t + \phi_1) + B \cos(\omega t + \phi_2) = C \cos(\omega t + \phi_3) \]

where

\[ C = \sqrt{(A \cos(\phi_1) + B \cos(\phi_2))^2 + (A \sin(\phi_1) + B \sin(\phi_2))^2} \]

and

\[ \phi_3 = \tan^{-1} \left( \frac{A \sin(\phi_1) + B \sin(\phi_2)}{A \cos(\phi_1) + B \cos(\phi_2)} \right) \]

It follows that:

\[ A \cos(\omega t) + B \sin(\omega t) = \left( \sqrt{A^2 + B^2} \right) \cos \left( \omega t + \tan^{-1} \left( \frac{-B}{A} \right) \right) \]

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**Preliminaries: Sinusoids**

Integrating a sinusoid over a full period yields 0. Where \( k \) is a positive integer:

\[ \int_{t_0}^{t_0+T} \sin \left( \frac{2\pi k t}{T} \right) dt = 0 \]

\[ \int_{t_0}^{t_0+T} \cos \left( \frac{2\pi k t}{T} \right) dt = 0 \]

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**Preliminaries: Sinusoids**

Harmonically-related sines and cosines are orthogonal. Where \( k \) and \( m \) are positive integers:

\[ \int_{t_0}^{t_0+T} \sin \left( \frac{2\pi k t}{T} \right) \cos \left( \frac{2\pi m t}{T} \right) dt = 0 \]
Preliminaries: Sinusoids

Where \( k \) and \( m \) are positive integers:

\[
\int_{0}^{T} \cos \left( \frac{2\pi k}{T} t \right) \cos \left( \frac{2\pi m}{T} t \right) \, dt = \begin{cases} 
T/2 & \text{if } k = m, \\
0 & \text{otherwise}
\end{cases}
\]

Preliminaries: Sinusoids

Where \( k \) and \( m \) are positive integers:

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\int_{0}^{T} \sin \left( \frac{2\pi k}{T} t \right) \sin \left( \frac{2\pi m}{T} t \right) \, dt = \begin{cases} 
T/2 & \text{if } k = m, \\
0 & \text{otherwise}
\end{cases}
\]

Preliminaries

**Fourier series** are sums of harmonically-related sinusoids:

\[
f(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right)
\]

where \( \omega_0 \) represents the "fundamental frequency."

Basis functions:
Fourier Representations of Signals

What signals can be represented by sums of harmonic components?

Fourier Analysis

Fourier posited that any periodic signal could be represented by a sum of a particular set of harmonic sinusoids. For now, we’ll run with that assumption.

Building up a signal by summing sinusoids is referred to as synthesis. But we are often more interested in analysis: given a periodic signal (assumed to be a sum of weighted, harmonically-related sinusoids), how can we find the weights?

Assume $f(t)$ is periodic in $T$ and is composed of a weighted sum of harmonics of $w_0 = 2\pi/T$:  

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos \left( \frac{2\pi k}{T} t \right) + d_k \sin \left( \frac{2\pi k}{T} t \right) \right)$$

How do we find the weights?
The “DC” Term

For largely historical reasons, the constant \( c_0 \) is referred to as the “DC” term: it represents the constant offset of a signal (if any). Stated another way, it represents the average value of a signal over one period:

\[
c_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \, dt
\]

Computing the Other \( c_k \) Terms

We are assuming that our signal is comprised of harmonically-related sines and cosines. Here is an example signal:

\[
f(t) = c_1 \cos(\omega_0 t) + d_1 \sin(\omega_0 t) + c_3 \cos(3\omega_0 t) + d_5 \sin(5\omega_0 t)
\]

Let’s imagine we wanted to determine the coefficient \( c_3 \), associated with \( \cos(3\omega_0 t) \).

Our strategy will be to multiply \( f(t) \) by \( \cos(3\omega_0 t) \) and integrate the result over one period.

**Check Yourself:** How does this help us find \( c_3 \)?

Computing the Other Terms

In general, we can compute the coefficients:

\[
c_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos \left( \frac{2\pi k t}{T} \right) \, dt = \frac{2}{T} \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} f(t) \cos(k\omega_0 t) \, dt
\]

We can use similar logic to find the coefficients associated with the sine waves:

\[
d_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin \left( \frac{2\pi k t}{T} \right) \, dt = \frac{2}{T} \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} f(t) \sin(k\omega_0 t) \, dt
\]
Two Different Views

Time domain:

Frequency domain:

Moving Between Representations

Synthesis:

\[ f(t) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos \left( \frac{2\pi k}{T} t \right) + d_k \sin \left( \frac{2\pi k}{T} t \right) \right) \]

Analysis:

\[ c_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \, dt \]
\[ c_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos \left( \frac{2\pi k}{T} t \right) \, dt \]
\[ d_k = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin \left( \frac{2\pi k}{T} t \right) \, dt \]

Check Yourself!

What are the Fourier series coefficients associated with the following signal?

\[ f(t) = 0.8 \sin(6\pi t) - 0.3 \cos(6\pi t) + 0.75 \cos(12\pi t) \]

Hint: What is the fundamental period?

Hint 2: No integrals required!
**Check Yourself!**

What are the Fourier series coefficients associated with the following with the following signal?

\[ f(t) = \sin^2(2\pi t) + \sin(2\pi t) \]

*Hint:* \[ \sin^2(u) = \frac{1 - \cos(2u)}{2} \]

**Check Yourself!**

Let \( c_k \) and \( d_k \) be the Fourier series coefficients associated with the following signal:

Which of the following statements are true?

1. \( c_k = 0 \) for all \( k \)
2. \( d_k = 0 \) if \( k \) is even
3. \( |d_k| \) decreases like \( k^2 \)
4. there are an infinite number of non-zero \( d_k \)

**Gibbs Phenomenon**

Partial sums of Fourier series of discontinuous functions “ring” near discontinuities: Gibbs’ phenomenon.

It is possible to decrease (and even eliminate) the ringing by decreasing the magnitudes of the Fourier coefficients at higher frequencies.
Summary

Today, we introduced the **CT Fourier Series**, which is a way of representing signals as sums of sinusoidal components.

Recitation 2A: Practice with finding Fourier series

Lecture 2B: CT Fourier Series (Complex Exponential Form)