Signals

6.003 is about signal processing.

Abstractly, a signal is a function that conveys information.

Examples:
- medical (EKG, EEG, MRI, ...)
- speech signals
- music
- seismic signals
- images
- video

Signal Processing

Signal processing is about extracting meaningful information from signals, and/or manipulating information in signals to produce new signals.
Transforms

Signals are functions that convey information. Sometimes, though, the straightforward way of representing a signal may not expose important properties of the signal.

It is often useful to have multiple different ways of looking at a signal.

For example, consider the following two representations of a speech signal:

We call such an alternative view of a signal a **transform**.

In 6.003, we will focus primarily on a family of transforms referred to as **Fourier** transforms, where signals are represented as sums of sinusoids, for example:

$$f(t) = \sum_{k=0}^{\infty} \left( c_k \cos k\omega_0 t + d_k \sin k\omega_0 t \right)$$
6.003 Pedagogy

Combine **theory**
- analysis and synthesis of signals
- transforms, time- and frequency-domain analysis
- convolution and deconvolution
- filtering and noise reduction

with authentic, real-world **applications** in
- music
- images
- physics
- ...

**Goal:** applications are fun and demonstrate usefulness of theory, but also deepen understanding of the theory.

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6.003 Pedagogy

Our common goal in science/engineering endeavors is to
- **model** some aspect of the world
- **analyze** the model
- **interpret** the results to gain better understanding

Model Result
World New Understanding
make model
analyze (math, computation)
interpret results

We want to give you practice with all three of these steps!

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Course Mechanics

**Lecture:** TR 10-11am (1-190)

**Recitation:** TR 11am-12pm (3-370) or WF 12-1pm (36-112)

**Problem Sets:** Tue - Tue, Lab checkoff due Friday
- **Drills:** (optional) practice with facts, definitions, etc
- **Exercises:** exam-style questions
- **Lab:** real-world applications

**Office Hours:**
- Friday 1-3pm (32-044)
- Wednesday, Thursday, and Sunday 7-9pm (34-501)

**Two quizzes** and a **final exam.**
Purpose

With one exception (the exams), everything we do is intended as a learning exercise, not as a test. Trying different approaches, struggling, and asking for help are all a normal part of the learning process!

Building a new skill takes time, practice, patience, and a willingness to seek out help. Don’t get discouraged if things are difficult; we are here to help!

Your Feedback is Important!

6.003 is still a work in progress!

Our goal is to present the course material in a way that encourages a deep understanding of the subject matter, while simultaneously being fun and engaging. We need your help and your feedback in order to make that happen.

6.003 Web Site

Just about everything in 6.003 happens via the web site:

http://mit.edu/6.003
Getting the Most out of 6.003

Lectures/Recitations:
- Step 1: Come to lecture/recitation!
- Take notes in your own words and review them later
- Ask questions! We want to have a conversation

PSets/Labs
- Start early!
- Look for connections to other problems, to lecture, to recitation.
- Ask questions!

Signal Properties: CT and DT
Continuous “time” (CT) vs discrete “time” (DT)

Physical signals are often of continuous domain:
- continuous time (in seconds)
- continuous spatial coordinates (in meters)

Computations manipulate functions of discrete domain:
- discrete time (in samples)
- discrete spatial coordinates (in samples)

Signal Properties: Periodicity
Periodic signals consist of repeated cycles (periods):

\[ x(t) = x(t + T) \]
\[ x[n] = x[n + N] \]

Periodic
aperiodic
**Signal Properties**

**Right-sided** signals are zero before some starting time. **Left-sided** signals are zero after some ending time.

![Diagram of right-sided and left-sided signals](image)

**Signal Properties: Boundedness**

The minima and maxima of **bounded** signals are finite.

![Diagram of bounded and unbounded signals](image)

**Signal Properties: Symmetry**

Signals can be **symmetric** or **antisymmetric** functions of time.

![Diagram of symmetric and antisymmetric signals](image)
**Signals: Notation**

Sometimes, we will want to talk about an entire signal as a single entity. Other times, we will want to talk about individual values of the signal. In 6.003, we will use the following notation to differentiate these cases:

1D, CT, whole signal: \( f(\cdot) \)
1D, CT, single value: \( f(t) \)

2D, CT, whole signal: \( f(\cdot,\cdot) \)
2D, CT, single value: \( f(x,y) \)

1D, DT, whole signal: \( f[\cdot] \)
1D, DT, single sample: \( f[n] \)

2D, DT, whole signal: \( f[\cdot,\cdot] \)
2D, DT, single sample: \( f[r,c] \)

**Sinusoids**

We’ll focus on sinusoids, since sinusoids are prevalent in nature (and human perception), and because they have some nice mathematical properties.

In general, a CT sinusoid has the form:

\[ x(t) = A \cos(\omega t + \phi) \]

- \( A \) is referred to as the amplitude
- \( \omega \) is referred to as the radian frequency
- \( \phi \) is referred to as a phase offset

**Check Yourself: Frequency**

Consider \( f(t) = \cos(\omega_1 t) \), shown below:

What is the value of \( \omega_1 \)? What are its units?
What would a graph of \( \cos(1.5\omega_1 t) \) look like?
Check Yourself: Phase

For some small, positive value \( a < \pi \), one of the following represents \( \cos(\omega_1 t + a) \), and the other represents \( \cos(\omega_1 t - a) \).

What is the value of \( a \)?

In which graph have we added \( a \), versus subtracting it?

Check Yourself: Sine

In general, a CT sinusoid has the form:

\[
x(t) = A \cos(\omega t + \phi)
\]

Can we represent \( \sin(\omega t) \) in this form?

If so, how? If not, why not?

DT Sinusoids

In general, a DT sinusoid has the form:

\[
x[n] = A \cos(\Omega n + \phi)
\]

- \( A \) is referred to as the amplitude
- \( \Omega \) is referred to as the discrete frequency
- \( \phi \) is referred to as a phase offset

Importantly, \( n \) is always an integer!
**Check Yourself: Frequency**

The following graph represents $x[n] = \cos(\Omega n)$ for some value of $\Omega$.

What is the value of $\Omega$? What are its units?

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**Aliasing**

Because we only have values at integer multiples of $\Omega$, there are multiple $\Omega$ values that lead to the exact same set of discrete points!

This graph could have resulted from any of an infinite number of different $\Omega$ values! We refer to this phenomenon as **aliasing**: the same signal can be described by different “names” (or aliases).

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**Aliasing**

In our example, we had:

- $x_1[n] = \cos(0.2\pi n)$
- $x_2[n] = \cos(2.2\pi n)$
- $x_3[n] = \cos(4.2\pi n)$

These all represent the exact same signal! They are all aliases for that signal.
**Aliasing**

Although there are multiple frequencies $\Omega$ that we could use to refer to this signal, it is difficult to “see” anything but $\Omega = 0.2\pi$ by looking at this graph.

We can remove the ambiguity of which frequency is represented by a set of samples by choosing the one in the range $0 \leq \Omega \leq \pi$.

We call that range of frequencies the **base band** of frequencies, and the value of $\Omega$ that falls in that range is often referred to as the **principal alias**.

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**Check Yourself: Aliasing**

Consider the following signal:

$$x[n] = \cos\left(\frac{5\pi}{4}\right)$$

Is it possible to represent this as a sinusoid with a discrete frequency in the range $0 \leq \Omega \leq \pi$?

If so, what is that value?
If not, why not?

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**Maximum Frequency**

If we limit our attention to frequencies in the base band, then there is a maximum possible discrete frequency $\Omega_{\text{max}} = \pi$.

Note this difference from CT, where there is no maximum frequency.
Relating CT and DT signals

Our goal is to develop signal processing tools that help us understand and manipulate the world.

\[
\begin{array}{c}
\text{Model} & \text{analyze (math, computation)} & \text{Result} \\
\text{make model} & \text{interpret results} & \text{New Understanding}
\end{array}
\]

The increasing power and decreasing cost of computation makes the use of computation increasingly attractive. However, many important signals are naturally described with continuous domain.

Sampling

We convert CT signals to DT signals by sampling.

\[
x(t) \quad x[n] = x(nT)
\]

\[
T \text{ (seconds / sample)} = \text{sampling interval} \\
f_s \text{ (samples / second)} = \text{sampling rate}
\]

We want to understand how sampling affects the information in a signal.

Sampling

We would like to sample in a way that preserves information. However, information is clearly lost in the sampling process.

\[
x(t)
\]

Clearly, we have no information about \(x(t)\) between the samples. Worse, information can be distorted through this sampling process!
Example

Example: Consider sampling a popular song at the following sampling rates:

- $f_s = 44100$ Hz
- $f_s = 22050$ Hz
- $f_s = 11025$ Hz
- $f_s = 5512$ Hz
- $f_s = 2756$ Hz
- $f_s = 1378$ Hz
- $f_s = 689$ Hz
- $f_s = 344$ Hz

Check Yourself!

What causes these distortions?

Relating CT and DT Frequencies

If a CT signal $x(t) = \cos \omega t$ is sampled at times $t = n/f_s$, the resulting DT signal is $x[n] = \cos(\Omega n)$ where

$$\Omega = \frac{\omega}{f_s}$$

If we restrict DT frequencies to the range $0 \leq \Omega \leq \pi$, then the corresponding CT frequencies are in the range $0 \leq \omega \leq \omega_N$ where

$$\omega_N = \pi f_s$$

This also restricts $f$ (Hz) to be in the range $0 \leq f \leq f_N$, where

$$f_N = \frac{\omega_N}{2\pi} = \frac{1}{2} f_s$$

$f_N$ (maximum frequency we can faithfully represent) is called the Nyquist frequency.
**Effects of Sampling**

If there are frequencies in the CT signal that are greater than the Nyquist frequency, they will alias to frequencies in the base band (that really don’t have anything to do with the original frequency!).

This is the main cause of the distortions from earlier.

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**Today’s Recitation**

Today in recitation: more with sampling and aliasing!