6.003 Final Exam  
Fall 2019

Name: [Answers]

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This exam is closed book, but you may use three 8.5 \times 11 sheets of paper. You may NOT use any electronic devices (including calculators, phones, etc).

If you have questions, please come to us at the front to ask them.

Please enter all solutions in the boxes provided. Extra work may be taken into account when assigning partial credit, but only work on pages with QR codes will be considered.

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**Question 1:** 16 Points

**Question 2:** 16 Points

**Question 3:** 24 Points

**Question 4:** 14 Points

**Question 5:** 16 Points

**Question 6:** 14 Points

**Total:** 100 Points
1 Transforms (16 Points)

1. Find the Fourier series coefficients of the signal \( x_1(\cdot) \), analyzed with \( T \) chosen to be the fundamental period of \( x_1(\cdot) \).

\[
x_1(t) = 2 \cos \left( \frac{\pi}{2} t \right) + 4 \cos \left( \frac{\pi}{3} t \right)
\]

In the box below, write a simple, closed-form answer for \( X_1[k] \):

\[
X_1[k] = \delta[k + 3] + 2\delta[k + 2] + 2\delta[k - 2] + \delta[k - 3]
\]

2. Find the Fourier series coefficients of the signal \( x_2[\cdot] \), shown below, which is periodic in \( N = 10 \).

In the box below, write a simple, closed-form answer for \( X_2[k] \):

\[
X_2[k] = \frac{1}{10} \left( 1 + e^{-j \frac{2\pi}{10} k} \right)
\]
3. Find the Fourier transform of the signal $x_3(\cdot)$ as defined below:

$$x_3(t) = \begin{cases} 
1 & \text{if } -1 \leq t \leq 2 \\
0 & \text{otherwise}
\end{cases}$$

In the box below, write a simple, closed-form answer for $X_3(\omega)$:

$$X_3(\omega) = \left( \frac{2 \sin \left( \frac{3}{2} \omega \right)}{\omega} \right) e^{-j \frac{\omega}{2}}$$

4. Find the Fourier transform of the signal $x_4[\cdot]$, defined below:

$$x_4[n] = \delta[n + 3] + \delta[n + 1] - \delta[n - 1] + \delta[n - 3]$$

In the box below, write a simple, closed-form answer for $X_4(\Omega)$:

$$X_4(\Omega) = 2 \cos(3\Omega) + 2j \sin(\Omega)$$
2 2D Convolution (16 Points)

For this problem, we will consider the following 2D signals, labeled $x_0$ through $x_{10}$, each of which is 9 rows $\times$ 13 columns. Note that the color scale is different between some of the signals.

For each circular convolution below, indicate which of the graphs on the facing page most closely matches the result by entering a single letter in each box. Note that, for each graph on the facing page, black corresponds to the lowest value in the signal (not necessarily 0), and white corresponds to the highest value in the signal (not necessarily 1).

$\begin{align*}
x_1 \ast x_0 & \quad F \\
x_2 \ast x_0 & \quad D \\
x_3 \ast x_0 & \quad L \\
x_4 \ast x_0 & \quad C \\
x_5 \ast x_0 & \quad G \\
x_6 \ast x_0 & \quad K \\
x_7 \ast x_0 & \quad I \\
x_8 \ast x_0 & \quad A \\
x_9 \ast x_0 & \quad E \\
x_{10} \ast x_0 & \quad J
\end{align*}$
3 Filtering (24 Points)

3.1 Part 1

Sketch the magnitude and phase of the frequency response of a linear, time-invariant system with the following unit-sample response:

\[ h_1[n] = \delta[n] + \delta[n - 1] \]

Label all important magnitudes, angles, and frequencies.
Worksheet (intentionally blank)
3.2 Part 2

Consider an LTI system with a unit sample response $h_2[\cdot]$ given by:

$$h_2[n] = \frac{1}{2} \delta[n] - \frac{1}{2} \delta[n - 2]$$

For each of the input signals $x_i[\cdot]$ below, assume that the response of the system to that input is given by $y_i[\cdot]$. For each, is it possible to represent $y_i[n]$ as a single pure sinusoid of the form $A_i \cos(\Omega_i n + \phi_i)$? If so, specify the appropriate values of $A_i$, $\Omega_i$, and $\phi_i$ by entering a single number in each box (square roots, $\pi$, and fractions are all OK). Otherwise, write `none` in all three boxes. If a box is irrelevant, write `any` in that box.

If $x_1[n] = 3$, is $y_1$ expressible as $y_1[n] = A_1 \cos(\Omega_1 n + \phi_1)$?

- $A_1 = 0$
- $\Omega_1 = \text{any}$
- $\phi_1 = \text{any}$

If $x_2[n] = \cos(\pi n/2)$, is $y_2$ expressible as $y_2[n] = A_2 \cos(\Omega_2 n + \phi_2)$?

- $A_2 = 1$
- $\Omega_2 = \pi/2$
- $\phi_2 = 0$

If $x_3[n] = \cos(4\pi n/3)$, is $y_3$ expressible as $y_3[n] = A_3 \cos(\Omega_3 n + \phi_3)$?

- $A_3 = \sqrt{3}/2$
- $\Omega_3 = 4\pi/3$
- $\phi_3 = \pi/6$

If $x_4[n] = \sin(\pi n/3) + \cos(\pi n/3)$, is $y_4$ expressible as $y_4[n] = A_4 \cos(\Omega_4 n + \phi_4)$?

- $A_4 = \sqrt{6}/2$
- $\Omega_4 = \pi/3$
- $\phi_4 = -\pi/12$

If $x_5[n] = (-1)^n$, is $y_5$ expressible as $y_5[n] = A_5 \cos(\Omega_5 n + \phi_5)$?

- $A_5 = 0$
- $\Omega_5 = \text{any}$
- $\phi_5 = \text{any}$
Worksheet (intentionally blank)
3.3 Part 3

Let \( x_0[n] = \delta[n + 1] + \delta[n] + \delta[n - 1] \), and let \( x_1[:] \) be a scaled and periodically-extended version of \( x_0[:] \), with repetitions every \( N_0 \) samples:

\[
x_1[n] = \sum_{m=-\infty}^{\infty} A x_0[n - m N_0] = \sum_{m=-\infty}^{\infty} A \delta[n - m N_0 + 1] + A \delta[n - m N_0] + A \delta[n - m N_0 - 1]
\]

As an example of the general shape of this function, here is an example with \( N_0 = 17 \) (though you should not assume that \( N_0 = 17 \) throughout the problem):

![Graph of \( x_1[n] \)]

Also let \( x_2[n] = B \cos(\Omega_0 n) \) for some value \( \Omega_0 \), and let \( x_3[n] = x_1[n] + x_2[n] \).

Each of the plots on the facing page shows the (purely real) DTFT of some function. Which of the graphs (1-8) corresponds to \( X_3(\Omega) \), the DTFT of \( x_3[n] = x_1[n] + x_2[n] \)? And what are the values of \( A, N_0, B, \) and \( \Omega_0 \)? Enter a single number in each box below:

<table>
<thead>
<tr>
<th>Graph number:</th>
<th>5</th>
</tr>
</thead>
</table>

\[
A = \frac{8}{3\pi} \\
N_0 = 8 \\
B = \frac{2}{\pi} \\
\Omega_0 = \frac{\pi}{3}
\]
4 DFT (14 Points)

For both parts of this problem, we’ll consider a signal $x[n]$, whose values are shown in the table below:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
<th>$11$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n]$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>$-1$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>$-2$</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

You may assume that $x[n] = 0$ for all values of $n$ not explicitly represented in the table above.

4.1 Part 1

Let $X(\cdot)$ represent this function’s DTFT, and let $R[k] = X\left(\frac{2\pi k}{4}\right)$.

Fill in the table below with the values of $r[n]$, the inverse DFT of $R[k]$ (computed with $N = 4$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
<th>$11$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r[n]$</td>
<td>24</td>
<td>$-4$</td>
<td>$-8$</td>
<td>4</td>
<td>24</td>
<td>$-4$</td>
<td>$-8$</td>
<td>4</td>
<td>24</td>
<td>$-4$</td>
<td>$-8$</td>
<td>4</td>
<td>24</td>
<td>$-4$</td>
<td>$-8$</td>
</tr>
</tbody>
</table>

4.2 Part 2

Let $X[\cdot]$ represent the DFT of $x[\cdot]$ (computed with $N = 9$), and let $H[\cdot]$ be the DFT of $h[n] = \delta[n] - \delta[n - 4]$ (also computed with $N = 9$).

If we define $Y[\cdot]$ such that $Y[k] = 9 \times X[k] \times H[k]$, fill in the table below with the values of $y[n]$, the inverse DFT of $Y[k]$ (also computed with $N = 9$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
<th>$11$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y[n]$</td>
<td>$-2$</td>
<td>$-3$</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$-2$</td>
<td>1</td>
<td>$-2$</td>
<td>$-3$</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Worksheet (intentionally blank)
5 MRI (16 Points)

Recall that the measurements made by MRI machines are samples of the DFT $X[k_r, k_c]$ of some underlying image $x[k_r, k_c]$, which we recover via an inverse DFT. Recall also that, unlike many signals we have considered, this image’s spatial domain representation is generally complex-valued.

Throughout this problem, we will consider the same $256 \times 256$ example image from lecture, for which $|x[k_r, k_c]|$ is shown below:

As we discussed in lecture and recitation, an important ongoing area of research involves attempting to faithfully reconstruct an image using as few samples of $X[k_r, k_c]$ as possible. In this problem, we will consider several different attempts to reduce the scan time of the image above.

5.1 Part 1

Consider reducing the scanning time by only sampling half of the rows of $X[k_r, k_c]$, creating a new image whose DFT is given by:

$$X_2[k_r, k_c] = \begin{cases} X[k_r, k_c] & \text{if } k_r \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Which image on the facing page most closely matches $|x_2[k_r, k_c]|$, the magnitude of the spatial domain representation of this image?

Enter a single letter: N

Also consider a different approach, where we still only sample half of the rows, but we fill in the missing rows via linear interpolation rather than leaving them as 0’s:

$$X_3[k_r, k_c] = \begin{cases} X[k_r, k_c] & \text{if } k_r \text{ is odd} \\ X[k_r + 1, k_c]/2 + X[k_r - 1, k_c]/2 & \text{otherwise} \end{cases}$$

Which image on the facing page most closely matches $|x_3[k_r, k_c]|$, the magnitude of the spatial domain representation of this image?

Enter a single letter: S
5.2 Part 2

Ben Bitdiddle suggests that a better way to cut down on scanning time would be to sample only half of the rows, but, particularly, to sample only where $0 \leq k_r \leq 128$, and then to use the conjugate symmetry of the DFT to fill in the missing values, i.e., for $-127 \leq k_r < 0$, set $X[k_r, k_c] = X^*[−k_r, −k_c]$.

Ben asserts that this approach will allow him to reconstruct $x[·, ·]$ exactly, while only explicitly sampling half of the DFT coefficients.

Is Ben’s assertion true? **Yes** or **No**: **No**

Briefly explain your reasoning:

The assumption that the DFT $X[·, ·]$ is conjugate symmetric is true if any only if the spatial-domain representation of the image $x[·, ·]$ is purely real.

So this approach would work if $x[·, ·]$ were real (regardless of the shape of $x[·, ·]$). However, because $x[·, ·]$ is actually complex valued, its DFT will not have this conjugate symmetry, and so Ben’s approach will not properly reconstruct the image.
Worksheet (intentionally blank)
6 Modulation (14 Points)

Consider the following modulation scheme, where $\omega_c >> \omega_m$.

\[
\begin{align*}
\cos\left(\frac{1}{2}\omega_m t\right) & \quad \cos\left(\left(\omega_c + \frac{1}{2}\omega_m\right) t\right) \\
x(t) \quad \text{LPF} \quad a(t) \quad + \quad y(t) \\
\sin\left(\frac{1}{2}\omega_m t\right) & \quad \sin\left(\left(\omega_c + \frac{1}{2}\omega_m\right) t\right) \\
\text{LPF} \quad b(t) \quad + 
\end{align*}
\]

Assume that each lowpass filter (LPF) is ideal, with cutoff frequency $\omega_m/2$, and note that $a(\cdot)$ and $b(\cdot)$ represent the respective outputs of the two low-pass filters.

Also assume that the input signal has the following (purely real) Fourier transform:

\[
X(\omega) = 1 \quad -\omega_m \quad \omega_m
\]

On the facing page, sketch the real and imaginary parts of $A(\omega)$, $B(\omega)$, and $Y(\omega)$. Label all important magnitudes and frequencies.
\[ \text{Re} (A(\omega)) \]

\[ \text{Im} (A(\omega)) \]

\[ \text{Re} (B(\omega)) \]

\[ \text{Im} (B(\omega)) \]

\[ \text{Re} (Y(\omega)) \]

\[ \text{Im} (Y(\omega)) \]
Worksheet (intentionally blank)