6.003 (Spring 2018)

Final Exam
May 21, 2018

Name:
Kerberos Username:

Enter all answers in the boxes provided.
Other work may be considered when assigning partial credit.

You have three hours.
This quiz is closed book, but you may use three 8.5 × 11 sheets of paper (both sides).
No calculators, computers, cell phones, music players, or other electronic devices.
1. Periodicities [10 points]

Part 1. Let $x_1[n]$ represent a discrete-time signal that is periodic in $n$ with period 12, as shown below.

Determine the Fourier series coefficients $a_k$ associated with this signal assuming that the periodicity $N$ is 12.

\[
\begin{align*}
a_0 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_1 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_2 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_3 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_4 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_5 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_6 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_7 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_8 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_9 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_{10} &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
a_{11} &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}}
\end{align*}
\]

Part 2. Let $x_2[n]$ represent a discrete-time signal that is periodic in $n$ with period 4, as shown below.

Determine the Fourier series coefficients $b_k$ associated with this signal assuming that the periodicity $N$ is 4.

\[
\begin{align*}
b_0 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
b_1 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
b_2 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \\
b_3 &= \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}} \phantom{\text{X}}
\end{align*}
\]
**Part 3.** Ben Bitdiddle notices that, for this particular set of signals, \( a_{3k} = b_k \) for all integer values of \( k \). Ben hypothesizes that this is true of all signals that are periodic in 4 when they are analyzed with \( N = 12 \).

Is Ben’s hypothesis correct? (‘Yes’ or ‘No’)

Briefly justify your answer (1-3 sentences).
2. Squares  [20 points]

Panels A-F illustrate six 2D discrete-time signals. Each signal has 48 rows and 48 columns. Black represents 0 and white represents 1. The origin of each of these panels is in the center of the panel.

For each signal, determine which of the following panels represents the magnitude of the $(48 \times 48)$ DFT of that signal, where black represents 0 and whiteness is proportional to the magnitude. The origin of each of these panels is in the center of the panel.

Enter your answers in the boxes below.

A: [ ]  B: [ ]  C: [ ]
D: [ ]  E: [ ]  F: [ ]
3. Programming  [10 points]

Consider the following Python function, designed to compute the result of convolving two 1-D signals. In this code, each signal is represented as a Python list of numbers, where the first value in the list represents the value of the signal at time \( n = 0 \). Values not explicitly represented in the lists are assumed to be 0.

01 | def convolve(x, h):
02 |     # first, flip the kernel
03 |     h = h[::-1]
04 |
05 |     # initialize the output
06 |     out = [0] * (len(x)+len(h)-1)
07 |
08 |     # loop over samples, applying superposition
09 |     for i in range(len(x)):
10 |         for j in range(len(h)):
11 |             out[i+j] += h[j] * x[i]
12 |     return out

This code is either correct, or close enough to correct that it can be made correct by changing at most 2 lines of code.

Is the code correct as written? ("Yes" or 'No"): 

If the code is incorrect, indicate the line(s) that should be changed, and what they should be replaced with. If only one change is needed, enter 'None' for the second line number.

<table>
<thead>
<tr>
<th>Line number</th>
<th>Should read</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4. Convolutions  [20 points]

Part 1

Let $x_1$ represent the discrete-time signal given by the following expression.

$$x_1[n] = 0.9^n u[n]$$

The result of convolving $x_1$ with an unknown signal $h_1$ is the unit sample signal,

$$(x_1 * h_1)[n] = \delta[n]$$

as illustrated in the following figure.

Determine an expression (not necessarily in closed form) for $h_1[n]$.

$$h_1[n] = \boxed{\text{expression}}$$

On the axes below, plot the magnitude and phase of $H_1(\Omega)$. In each sketch, label all important values.

| $|H_1(\Omega)|$ | $\angle H_1(\Omega)$ |
|-----------------|--------------------------|
| $\Omega$       | $\Omega$                |
Part 2

Let $x_2$ represent the discrete-time signal given by the following expression.

$$x_2[n] = \delta[n] - 0.8\delta[n - 2]$$

The result of convolving $x_2$ with an unknown signal $h_2$ produces a new signal ($y_2[n]$)

$$(x_2 \ast h_2)[n] = y_2[n] = \begin{cases} (-0.8)^{n/2}u[n] & \text{for } n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

as illustrated below.

Determine an expression (not necessarily in closed form) for $h_2[n]$.

$$h_2[n] =$$

On the axes below, plot the magnitude and phase of $H_2(\Omega)$. In each sketch, label all important values.

$|H_2(\Omega)|$  \hspace{2cm}  $\angle H_2(\Omega)$
5. Shapes  

[20 points]

Each of the images below was created by computing the inverse DFT on a 24-by-24 array of DFT coefficients, of which at most 5 were non-zero. In each of these images, black represents a value of 0, and white represents a value of 1. The origin of each image is in its upper left corner.

For each image, enter the locations \((k_x, k_y)\) of all non-zero values in the associated DFT. If the image could not have been made from an array of the form described above, enter **None** in the box instead.

Enter your answers on the facing page.
What are the locations of the nonzero values in the DFT associated with image A?

What are the locations of the nonzero values in the DFT associated with image B?

What are the locations of the nonzero values in the DFT associated with image C?

What are the locations of the nonzero values in the DFT associated with image D?

What are the locations of the nonzero values in the DFT associated with image E?

What are the locations of the nonzero values in the DFT associated with image F?

What are the locations of the nonzero values in the DFT associated with image G?

What are the locations of the nonzero values in the DFT associated with image H?

What are the locations of the nonzero values in the DFT associated with image I?

What are the locations of the nonzero values in the DFT associated with image J?

What are the locations of the nonzero values in the DFT associated with image K?

What are the locations of the nonzero values in the DFT associated with image L?
6. Bitdiddle Returns [20 points]

Ben Bitdiddle took a photograph of his cat, but he only saved the associated DFT coefficients $X[k_x, k_y]$, rather than saving the original image. However, he knows the original 51x77 image looked like this:

Ben tries several different methods of recovering the original image based on $X[k_x, k_y]$. For each of the methods described below, indicate which of the images from the facing page (A-T) would have resulted from that approach. In these images, grey colors represent positive values (black represents 0, white represents 1), and red represents negative values (black represents 0, bright red represents -1).

If an approach would have led to an image with nonzero imaginary components (and thus would have resulted in a Python error when trying to save the image), put an X in that box instead.

1. Applying the iDFT to the real part of $X$.
2. Applying the iDFT to the imaginary part of $X$.
3. Applying the iDFT to $j$ times the imaginary part of $X$.
4. Applying the iDFT to $X$ after setting $X[0, 0] = 0$.
5. Applying the iDFT to $X$ after setting $X[25, 38] = 0$.
6. Applying the iDFT to $X$ after subtracting $1/(51 \times 77)$ from every value.
7. Applying the iDFT to $X$ after multiplying every value by $e^{j\pi}$.
8. Applying the iDFT to $X$ after multiplying every value except $X[0, 0]$ by $e^{j\pi}$.
9. Applying the iDFT to $X$ after negating the phase of every value.

Note that the imaginary part of a number $a + bj$ is $b$.

For each of these approaches, which image on the facing page matches the result? Enter an X if a Python error would occur.

1:  
2:  
3:  
4:  
5:  
6:  
7:  
8:  
9:  

A: 
B: 
C: 
D: 
E: 
F: 
G: 
H: 
I: 
J: 
K: 
L: 
M: 
N: 
O: 
P: 
Q: 
R: 
S: 
T: