Practice Quiz #1

Name:
Kerberos Username:

Enter all answers in the boxes provided.

You have two hours.
This quiz is closed book, but you may use one 8.5 × 11 sheet of paper (two sides).
No calculators, computers, cell phones, music players, or other electronic devices.
1. Sum of Signals  \([n \text{ points}]\)

Let \(x[n]\) represent the sum of two discrete-time signals:

\[
x[n] = x_1[n] + x_2[n].
\]

The signal \(x_1[n]\) is periodic with a period of 4.

\[
x_1[n] = x_1[n + 4] \quad \text{for} \quad n \in \mathbb{Z}
\]

The signal \(x_2[n]\) is represented by its Fourier series coefficients \(X_2[k]\), which are periodic with a period of 5.

\[
X_2[k] = X_2[k + 5]
\]

Find \(x[-17]\).

\[
x[-17] = 1 + \cos \frac{2\pi}{5}
\]
2. Find the Magnitude \([n \text{ points}]\)

Consider the following plots of the magnitudes of the Fourier series coefficients for four discrete-time signals, that are each periodic in \(N = 20\).

For each of the following signals, determine which if any of the previous plots shows the magnitude of its Fourier series coefficients.

- \(x_1[n]\)
  - Enter A to D or None: D

- \(x_2[n]\)
  - Enter A to D or None: B

- \(x_3[n]\)
  - Enter A to D or None: A

- \(x_4[n]\)
  - Enter A to D or None: None
3. Manipulating Fourier Series \([n \text{ points}]\)

**Part a.** Let \(x_1(t)\) represent a continuous-time periodic signal with a period of \(T\) and Fourier coefficients \(X_1[k]\). Find an expression for the Fourier coefficients \(X_2[k]\) of the signal
\[
x_2(t) = x_1(t - T/2)
\]

\[
X_2[k] = e^{-j\pi k} X_1[k]
\]

\[
X_1[k] = \frac{1}{T} \int_T x(t)e^{-j\frac{2\pi}{T}kt} dt
\]

\[
X_2[k] = \frac{1}{T} \int_T y(t)e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T} \int_T x(t - T/2)e^{-j\frac{2\pi}{T}kt} dt
\]

Let \(\tau = t - T/2\) then

\[
X_2[k] = \frac{1}{T} \int_T x(\tau)e^{-j\frac{2\pi}{T}k(\tau+T/2)} d\tau = e^{-j\frac{2\pi}{T}kT/2} \frac{1}{T} \int_T x(\tau)e^{-j\frac{2\pi}{T}k\tau} d\tau = e^{-j\pi k} X_1[k]
\]

**Part b.** Let \(x_3[n]\) represent a discrete-time periodic signal with a period of \(N\) and Fourier coefficients \(X_3[k]\), where \(N\) is an even integer. Let \(x_4[n]\) represent the signal whose Fourier series coefficients are
\[
X_4[k] = e^{-j\pi k} X_3[k]
\]

Find an expression for \(x_4[n]\) in terms of \(x_3[n]\).

\[
x_4[n] = x_3[n - N/2]
\]

\[
x_3[n] = \sum_{k=\langle N \rangle} X_3[k] e^{j\frac{2\pi}{N}kn}
\]

\[
x_4[n] = \sum_{k=\langle N \rangle} X_4[k] e^{j\frac{2\pi}{N}kn}
= \sum_{k=\langle N \rangle} e^{-j\pi k} X_3[k] e^{j\frac{2\pi}{N}kn}
= \sum_{k=\langle N \rangle} X_3[k] e^{j\frac{2\pi}{N}k(n - N/2)}
= x[n - N/2]
\]
4. **Find the Fundamental [n points]**

Let $x[n]$ represent a periodic, discrete-time signal with a period of 12 as shown below.

\[ x[n] = x[n + 12] \]

A new signal $y[n]$ is derived from $x[n]$ by inserting 2 zeros between adjacent samples of $x[n]$ so that the new sequence is given by

\[
y[n] = \begin{cases} 
  x[n/3] & \text{if } n \text{ is an integer multiple of 3} \\
  0 & \text{otherwise}
\end{cases}
\]

Since $y[n]$ is periodic, it can be described by a Fourier series of the following form

\[
y[n] = c_0 + \sum_{k=1}^{N/2} c_k \cos \frac{2\pi kn}{N} + \sum_{k=1}^{N/2} d_k \sin \frac{2\pi kn}{N}
\]

Find $c_1$, $d_1$, and the constant $N$.

\[
c_1 = 0
\]

\[
d_1 = \frac{1}{6}
\]

\[
N = 36
\]
The period of $y[n]$ is 3 times the period of $x[n]$. Thus $N = 36$.

Let $a_k$ represent the Fourier series coefficients of $y[n]$.

$$a_k = \frac{1}{36} \sum_{n=0}^{36} x[n] e^{-j\frac{2\pi}{36}nk} = \frac{1}{36} e^{-j\frac{2\pi}{36}3k} + \frac{2}{36} e^{-j\frac{2\pi}{36}9k} + \frac{1}{36} e^{-j\frac{2\pi}{36}15k}$$

$$a_1 = \frac{1}{36} \left( e^{-j\frac{2\pi}{36}} + 2e^{-j\frac{2\pi}{36}} + e^{-j\frac{2\pi}{36}} \right)$$

$$a_1 = \frac{1}{36} (e^{-j\frac{\pi}{6}} + 2e^{-j\frac{\pi}{6}} + e^{-j\frac{5\pi}{6}}) = -j \frac{1}{12}$$

$$a_{-1} = a_1^* = j \frac{1}{12}$$

$$a_1 e^{j\frac{2\pi}{36}} + a_{-1} e^{-j\frac{2\pi}{36}} = -j \frac{1}{12} \left( \cos \frac{2\pi n}{36} + j \sin \frac{2\pi n}{36} \right) + j \frac{1}{12} \left( \cos \frac{2\pi n}{36} - j \sin \frac{2\pi n}{36} \right)$$

$$= \frac{2}{12} \sin \frac{2\pi n}{36}$$

$c_1 = 0$

d_1 = \frac{1}{6}$
5. **Missing Components**  

Consider a set of discrete-time signals $x_i[n]$ that are each periodic such that

$$x_i[n] = x_i[n + 8]$$

for all values of $n$. Five of these signals are defined by their Fourier series coefficients calculated for a period of $N = 8$ and illustrated in the following figure.

Use the following panels to answer questions on the following page.
Which (if any) panel shows \( |x_1[n]| \)? Enter A to I or None: B

Which (if any) panel shows \( |x_2[n]| \)? Enter A to I or None: B

Which (if any) panel shows \( |x_3[n]| \)? Enter A to I or None: B

Which (if any) panel shows the real part of \( x_3[n] \)? Enter A to I or None: D

Which (if any) panel shows the real part of \( x_5[n] \)? Enter A to I or None: H

Which (if any) panel shows the imaginary part of \( x_2[n] \)? Enter A to I or None: C

Which (if any) panel shows the imaginary part of \( x_3[n] \)? Enter A to I or None: E

Which (if any) panel shows the imaginary part of \( x_4[n] \)? Enter A to I or None: E
6. Fourier Transform \([n \text{ points}]\)

Part 1

The continuous-time signal \(x_1(t)\) is defined by the following plot and is zero outside the indicated range of \(t\).

Compute \(X_1(\omega)\), the Fourier Transform of \(x_1(t)\).

\[
X_1(\omega) = 2\sin \frac{3\omega}{\omega} - 2\sin \frac{2\omega}{\omega}
\]

The signal \(x(t)\) can be written as the difference between a rectangular pulse that extends from \(-3\) to \(3\) and a rectangular pulse that extends from \(-2\) to \(2\). The Fourier transform of the former is

\[
\int_{-3}^{3} e^{-j\omega t} \, dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-3}^{3} = 2\sin \frac{3\omega}{\omega}
\]

Similarly, the Fourier transform of the latter is \(2\sin \frac{2\omega}{\omega}\). Thus the total answer is

\[
X(\omega) = 2\sin \frac{3\omega}{\omega} - 2\sin \frac{2\omega}{\omega}
\]
Part 2

Determine the Fourier transform of $x_2(t)$ given by the following expression

$$x_2(t) = \begin{cases} e^{\lvert t \rvert} & \text{if } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and illustrated below.

![Graph of $x(t)$](image)

Enter a closed-form expression for the Fourier transform in the box below, both in complex exponential form and in terms of sines and cosines.

$$X_2(\omega) = \frac{2e \cos \omega + 2e \omega \sin \omega - 2}{1 + \omega^2}$$

Calculate the Fourier transform of the right half of this function:

$$X_r(\omega) = \int_0^1 e^t e^{-j\omega t} \, dt = \int_0^1 e^{(1-j\omega)t} \, dt = \left. \frac{e^{(1-j\omega)t}}{1-j\omega} \right|_0^1 = \frac{e^{(1-j\omega)} - 1}{1-j\omega}$$

The Fourier transform of the left part is

$$X_l(\omega) = X_r(-\omega) = \frac{e^{(1+j\omega)} - 1}{1+j\omega}$$

Therefore

$$X(\omega) = X_r(\omega) + X_l(\omega) = \frac{e^{(1-j\omega)} - 1}{1-j\omega} + \frac{e^{(1+j\omega)} - 1}{1+j\omega} = \frac{2e \cos \omega + 2e \omega \sin \omega - 2}{1 + \omega^2}$$
Part 3

Consider a discrete-time function $x_3[n]$ given by the following expression:

$$x_3[n] = \begin{cases} 
-1 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
0 & \text{otherwise}
\end{cases}$$

Sketch the magnitude and angle of the Fourier transform of $x_3$, $X_3(\Omega)$, on the axes below. Label key points.

Straightforward application of the DTFT analysis equation yields:

$$X_3(\Omega) = \sum_{n=-\infty}^{\infty} x_3[n]e^{-j\Omega n} = e^{-j\Omega} - 1$$

To get a better sense for the shape of these curves, we can pull out a common factor to put this closer to a form we have seen before:

$$X_3(\Omega) = e^{-j\Omega} - 1 = e^{-j\Omega/2} (e^{-j\Omega/2} - e^{j\Omega/2}) = e^{-j\Omega/2} (-2j \sin(\Omega/2))$$

From here, it is easier to see that the phase is changing linearly with $\Omega$ (with the exception of the discontinuities at $\Omega = 0, 2\pi, 4\pi, \ldots$ where the sine wave changes signs), and that the magnitude is $2|\sin(\Omega/2)|$.

You may also find it helpful to first plot the magnitude and phase of $-2j \sin(\Omega/2)$, and then to think about how the $e^{-j\Omega/2}$ term affects those plots.
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