**6.003: Signal Processing**

**Correlation and Matched Filters**

**November 20, 2018**

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**Last Lab: Effects of Noise**

Adding imperceptible amounts of noise → enormous consequences. In the absence of noise, blurring was removed by inverse filtering.

![inverse filter](image1)

But imperceptible noise at the input dominated the output image.

![inverse filter](image2)

Fixed by making small changes to the inverse filter.

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**Noise is Everywhere**

Sources of noise.

**Thermal (Johnson) noise**: electronic noise caused by thermal agitations. Manifests as an additive noise with gaussian distribution.

**Quantization**: technically a distortion (since it’s not random) these errors result when we represent sample values with elements from a discrete set (such as 8-bit representations for conventional images).

**Shot (Poisson) noise**: variations in brightness caused by the particle nature of photons. If photons arise independently but at a given rate, then the number of photons collected over some time period varies in proportion to the square root of the average number.

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**Communications**

Noise considerations are of central importance in communications applications, where the goal is to move information.

Example: serial transmission of DNA sequence as follows:

\[
\begin{align*}
A: & x_1(t) \\
G: & x_2(t) \\
C: & x_3(t) \\
T: & x_4(t)
\end{align*}
\]

sequence: \( y(t) \)

What is the best way to decode if \( y(t) \) is corrupted by noise?

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**Check Yourself**

Assume that the decoder can be implemented with an LTI filter.

Each of the following impulse responses is 0 if \( t < 0 \) or \( t > 4 \). Which one maximizes \( y(4) \) for \( x(t) \) shown above?
Communications

Today's lecture provides some new concepts for thinking about communications systems.

Correlation

Cross-correlation measures the similarity of two signals.

Cross-Correlation:

\[(x \ast y)(t) \equiv \int_{-\infty}^{\infty} x(\tau)y(\tau + t) \, d\tau\]

combines parts of signals with the same time difference:

\[(\tau + t) - \tau = t\]

Similar to Convolution:

\[(x \ast y)(t) \equiv \int_{-\infty}^{\infty} x(\tau)y(t - \tau) \, d\tau\]

but convolution combines parts of signals with the same total lag:

\[(t - \tau) + \tau = t\]

Applications of Cross-Correlation

Large cross-correlation at \(t = t_p\) indicates similarity of \(x(t)\) and \(y(t + t_p)\). Therefore we can find a target signal \(x(t)\) in a noisy signal \(y(t)\) by finding the peak in \((x \ast y)(t)\):

\[(x \ast y)(t)\]

Relation Between Correlation and Convolution

Definitions.

\[(x \ast y)(t) = \int x(\tau)y(\tau + t) \, d\tau\]

\[(x \ast y)(t) = \int x(\tau)y(t - \tau) \, d\tau\]

Let \(x_f(t) = x(-t)\) represent a flipped version of \(x(t)\):

\[(x_f \ast y)(t) = \int x(-\tau)y(\tau + t) \, d\tau = \int x(\lambda)y(t - \lambda) \, d\lambda\]

\[= (x \ast y)(t)\]

Correlating \(x_f(t)\) with \(y(t)\) is the same as convolving \(x(t)\) and \(y(t)\).

Let \(y_f(t) = y(-t)\) represent a flipped version of \(y(t)\):

\[(x \ast y_f)(t) = \int x(\tau)y(-\tau - t) \, d\tau = (x \ast y)(-t)\]

Different from correlating \(x_f(t)\) with \(y(t)\); different from convolution.

Correlation is NOT Commutative

Commuting the order of the inputs flips the correlation.

\[(x \ast y)(t) = \int x(\tau)y(\tau + t) \, d\tau\]

\[(y \ast x)(t) = \int y(\tau)x(\tau + t) \, d\tau\]

\[= \int x(\lambda)y(\lambda - t) \, d\lambda\]

\[\text{where } \lambda = \tau + t\]

\[= (x \ast y)(-t)\]

Radar, Ultrasound, and Optical Tomography

Cross-correlation is widely used in range-finding applications.

Each of these images was created by correlating an input signal (radio wave, ultra-sound, or light) with its reflection off a structure.
Autocorrelation

Autocorrelation measures patterns within a signal.

Cross-Correlation:

\[(x \ast y)(t) \equiv \int_{-\infty}^{\infty} x(\tau) y(\tau + t) d\tau\]

Autocorrelation:

\[(x \ast x)(t) \equiv \int_{-\infty}^{\infty} x(\tau) x(\tau + t) d\tau\]

Energy

The autocorrelation at \(t = 0\) is equal to the energy in the signal.

\[(x \ast x)(t) = \int x(\tau) x(\tau + t) d\tau\]

The total energy of a continuous-time signal \(x(t)\) is defined as

\[E = \int x^2(t) dt\]

and that of a discrete-time signal \(x[n]\) is similarly

\[E = \sum x^2[n]\]

Neither of these definitions matches the conventional notion of energy in physics, but the concepts are closely related. For example, if \(x(t)\) represents the voltage across a 1\(\Omega\) resistor, then \(E\) equals the energy dissipated by the resistor.

Schwartz Inequality

Define the “inner product” of two functions as

\[\langle a, b \rangle = \int_0^T a(t) b(t) dt\]

Let \(\lambda = \frac{\langle h, h \rangle}{\langle a, a \rangle}\), where \(\langle h, h \rangle > 0\) if \(h(t) \neq 0\) (if \(h(t) = 0\) the proof is trivial).

Then

\[0 \leq (x - \lambda h, x - \lambda h)\]

\[= \langle x, x \rangle - 2\lambda \langle x, h \rangle + \lambda^2 \langle h, h \rangle\]

\[= \langle x, x \rangle - \langle (x, h) \rangle^2 \frac{\langle h, h \rangle}{\langle h, h \rangle}\]

and therefore

\[\langle (x, h) \rangle^2 \leq \langle x, x \rangle \langle h, h \rangle\]
Optimality of Matched Filtering

We can use the Schwartz inequality to show that a matched filter optimally detects \( x(t) \) from the output \( y(t) \) for \( 0 \leq t \leq T \), as follows.

If \( h(t) \) is the impulse response of the decoding system, then

\[
y(T) = (x \ast h)(T) = \int_0^T x(\tau)h(T - \tau)d\tau = \int_0^T x(\tau)h_f(\tau)d\tau = \langle x, h_f \rangle
\]

where \( h_f(t) = h(T - t) \). By the Schwartz inequality,

\[
(\langle x, h_f \rangle)^2 \leq \langle x, x \rangle \langle h_f, h_f \rangle.
\]

If the energy in \( h(t) \) is bounded so that \( \langle h, h \rangle = \langle h_f, h_f \rangle \leq \langle x, x \rangle \),

\[
(\langle x, h_f \rangle)^2 \leq \langle x, x \rangle \langle h_f, h_f \rangle \leq (\langle x, x \rangle)^2.
\]

For a matched filter, \( h(t) = x(T - t) \) and

\[
\langle x, h \rangle = (x \ast h)(T) = \int_0^T x(\tau)h(T - \tau)d\tau = \int_0^T x(\tau)x(\tau)d\tau = \langle x, x \rangle.
\]

Thus the matched filter achieves the Schwartz bound with equality. This means that no impulse response with energy less than that in \( x(t) \) has output \( y(T) \) that is bigger than that of a matched filter.

Decoding with a Matched Filter (why look at \( t = T \)?)

The simplest form of a matched filter has impulse response \( h(t) = x(-t) \).

Let \( y(t) = (x \ast h)(t) = \int x(\tau)h(t - \tau)d\tau = \int x(\tau)x(\tau - t)d\tau \)

Then, by the Schwartz inequality,

\[
\left( \int x(\tau)x(\tau - t)d\tau \right)^2 \leq \left( \int x(\tau)x(\tau)d\tau \right) \left( \int x(\tau - t)x(\tau - t)d\tau \right)
\]

\[
= \left( \int x(\tau)x(\tau)d\tau \right) \left( \int x(\tau)x(\tau)d\tau \right)
\]

\[
= \left( \int x(\tau)x(\tau)d\tau \right)^2
\]

Thus the maximum output of such a filter occurs at \( t = 0 \).

Shifting the impulse response by \( t = T \) (where \( T = \) duration of \( x(t) \))

\( h_c(t) = x(T - t) \)

makes it causal, and shifts the maximum output to \( t = T \).

Complex Signals

Correlation, energy, Parseval’s relation, and matched filters can be generalized to apply to complex-valued signals.

**Correlation**

\[
(x \ast y)(t) = \int x(\tau)y^*(\tau + t)d\tau
\]

**Energy**

\[
E = \int x(t)x^*(t)dt = \int |x(t)|^2dt
\]

**Parseval’s Relation**

\[
\int |x(t)|^2dt = \frac{1}{2\pi} \int |F(\omega)|^2d\omega
\]

**Matched Filter**

\( h(t) = x^*(T - t) \)

Summary

Correlation quantifies the similarity of signals.

- useful for quantifying time shifts (as in radar and tomography)

Autocorrelation quantifies patterns within a signal.

- useful for finding the period of a quasi-periodic signal

Matched filters are useful for finding a pattern that is specified by one signal in a second signal.

- useful in decoding information coded in a message