6.003: Signal Processing

Analyzing Time-Varying Signals

- Review of Discrete Fourier Transform (DFT)
- Windowing
- Short-time Fourier Transform (STFT)
- Spectrograms

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What is 6.003?

What is a signal?

Abstractly, a signal is a function that conveys information. Signal processing is about extracting meaningful information from signals, and/or manipulating information in signals to produce new signals.

What is a transform?

Provide multiple views/perspectives on a signal. Some information more clearly visible (and/or more easily manipulable) from one perspective than another.

Why Fourier?

One reason: Many aspects of human perception are related to frequency representation. Some things apparent in frequency but not in time (and vice versa).

Example: what are the following sounds, and how do they differ?

Why DFT?

So many transforms, why introduce another? Computationally feasible (opens doors to analyzing complicated signals).

Most modern signal processing is based on the DFT, and we’ll use the DFT almost exclusively moving forward in 6.003.

Sidenote: FFT (Fast Fourier Transform) is an algorithm for computing the DFT efficiently (pset 7).

Why Did We Bother With the Other Ones?

In part, because they inform our understanding/interpretation of the results of the DFT.

Our common goal with other science/engineering endeavors is to:

- Model some aspect of the world
- Analyze the model, and
- Interpret results to gain better understanding.

Model (computer) Result (human)

World New Understanding (human)

The computer is useless if we can’t interpret the results!

Discrete Fourier Transform

Definition and comparison to other Fourier representations.

<table>
<thead>
<tr>
<th></th>
<th>analysis</th>
<th>synthesis</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>DFT</strong></td>
<td>$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn}$</td>
<td>$x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi}{N} kn}$</td>
</tr>
<tr>
<td><strong>DTFS</strong></td>
<td>$X[k] = \frac{1}{N} \sum_{n=0(N)} x[n] e^{-j \frac{2\pi}{N} kn}$</td>
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</tr>
<tr>
<td><strong>DTFT</strong></td>
<td>$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \Omega n}$</td>
<td>$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j \Omega n} d\Omega$</td>
</tr>
</tbody>
</table>

DTFS: $x[n]$ is presumed to be periodic in $N$

DTFT: $x[n]$ is arbitrary

DFT: only a portion of an arbitrary $x[n]$ is considered
If a signal is periodic in the DFT analysis period $N$, then the DFT coefficients are equal to the DTFS coefficients. Let $x_1[n] = \cos \frac{2\pi n}{N}$. Then the DFT coefficients are

$$X_1[k] = \frac{1}{N} \sum_{n=0}^{N-1} x_1[n] e^{-j \frac{2\pi}{N} kn} = \frac{1}{2} \delta[k-1] + \frac{1}{2} \delta[k-63]$$

as plotted below.

The coefficients are the same as the Fourier series coefficients.

If a signal is not periodic in the DFT analysis period $N$, then there are no Fourier series coefficients to compare. Let $x_2[n] = \cos \frac{3\pi n}{N}$. Then the DFT coefficients are

$$X_2[k] = \frac{1}{N} \sum_{n=0}^{N-1} x_2[n] e^{-j \frac{2\pi}{N} kn}$$

as plotted below.

Even though $x_2[n]$ contains a single frequency $\Omega = 3\pi/64$, there are large components at every component $k$.

Although $x_2[n] = \cos \frac{3\pi n}{64}$ was not periodic in $N = 64$, we can define a signal $x_3[n]$ that is equal to $x_2[n]$ for $0 \leq n < 64$ and periodic in $N = 64$.

The DFT coefficients for this signal are the same as those for $x_2[n]$:

Furthermore, the DFT coefficients of $x_3[n]$ equal the DTFS coefficients of $x_3[n]$. The large number of non-zero coefficients are necessary to produce the step discontinuity at $n = 64$.

Graphical depiction of relation between DFT and DTFT.

Blurring degrades frequency resolution.

Important: The large number of non-zero coefficients is not just information about $x$! It is information about $x_w$ (which is affected by the window!).
**Time/Frequency Tradeoff**

Longer windows provide finer frequency resolution.

\[
\begin{align*}
\Omega \mid W(\Omega) \mid \pi \\
N = 16 \\
\Omega \mid W(\Omega) \mid \pi \\
N = 32 \\
\Omega \mid W(\Omega) \mid \pi \\
N = 64
\end{align*}
\]

However, window still has frequency content across entire spectrum.

**Windowing**

A large part of the problem is the step discontinuity that we see when considering the DTFS of a periodically-extended signal:

Let \( x[n] = \cos \frac{2\pi n}{N} \) \( x_r \) is a windowed, periodically-extended version of \( x \). The DTFS coefficients of \( x_r \) are the DFT coefficients of \( x \).

\[
\begin{align*}
\Omega \mid W(\Omega) \mid \pi \\
N = 16 \\
\Omega \mid W(\Omega) \mid \pi \\
N = 32 \\
\Omega \mid W(\Omega) \mid \pi \\
N = 64
\end{align*}
\]

Idea: modify the original signal so as to avoid creating that discontinuity.

**Windowing: Triangular Window**

Instead of multiplying \( x \) by a rectangular window, what happens if we instead multiply by a triangle (that slopes to 0 at times \( n = 0 \) and \( n = N - 1 \))?

\[
\begin{align*}
x_r[n] \\
|X_r[k]|
\end{align*}
\]

Far less smearing!

**Windowing: Hann Window**

Another option is a Hann (or “Hanning”) window:

\[
\begin{align*}
& N = 32 \\
& \log |W(\Omega)|
\end{align*}
\]

Shape in time affects shape in frequency.
Windowing

Regardless of the shape of the window, increasing $N$ improves frequency resolution. However, there is no such thing as a free lunch!

Time/Frequency Tradeoff

Longer windows provide finer frequency resolution.

However, longer windows provide less temporal resolution.

Example: 4 tunes

How to tell them apart?

→ fundamental tradeoff between resolution in frequency and time.

Short-time Fourier Transform

STFT is a compromise between time- and frequency-domain representations, representing the frequency content of the signal at various points in time.

Formally, we define the STFT of a signal $x$ as:

$$\text{STFT}\{x]\{m,k\} = \sum_{n=0}^{N-1} x[n + m \times s]w[n]e^{j2\pi kn/N}$$

Conceptually, we are taking the DFT of successive windowed regions of the original signal (and these regions may overlap).

Short-time Fourier Transform and Spectrograms

The STFT enhances our ability to reason about the frequency content of signals at various points in time. It is often visualized using a spectrogram, which is defined to be the magnitude squared of the STFT.

Time-Varying Signals

Real world signals (i.e., speech, music, images) often have frequency content that varies with time.

Problem with DFT: events that are local in time are global in frequency (and vice versa). Sudden changes and local variations can be difficult to detect.

Example: 4 tunes

How to tell them apart?