Discrete-Time Fourier Series

- orthogonality of harmonically related DT sinusoids
- DT Fourier series relations
- differences between CT and DT Fourier series
- properties of DT Fourier series
Express a periodic signal as a sum of harmonically related sinusoids

\[ x(t) = \sum_{k=0}^{\infty} \left( c_k \cos k\omega_o t + d_k \sin k\omega_o t \right) \]

where \( \omega_o \) represents the fundamental frequency.

Basis functions:
Continuous-Time Fourier Series

Separating harmonic components relies on two key observations.

1. Multiplying two harmonics produces a new harmonic with the same fundamental frequency:

\[ e^{jk\omega_0 t} \times e^{jl\omega_0 t} = e^{j(k+l)\omega_0 t}. \]

2. The integral of a harmonic over any time interval with length equal to the period \( T \) is zero unless the harmonic is at DC:

\[
\int_{t_0}^{t_0+T} e^{jk\omega_0 t} dt \equiv \int_{T} e^{jk\omega_0 t} dt = \begin{cases} T & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} = T\delta[k].
\]

→ Fourier components are orthogonal.
Continuous-Time Fourier Series

Assume that \( x(t) \) is periodic in \( T \) and is composed of a weighted sum of harmonics of \( \omega_0 = 2\pi/T \).

\[
x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}
\]

Then “sift” out one component:

\[
\int_T x(t) e^{-j\omega_0 lt} dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} e^{-j\omega_0 lt} dt
\]

\[
= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_0 (k-l)t} dt
\]

\[
= \sum_{k=-\infty}^{\infty} a_k T \delta[k - l] = a_l T
\]

Solving for \( a_l \) provides an explicit formula for the coefficients:

\[
a_k = \frac{1}{T} \int_T x(t) e^{-j\omega_0 kt} dt = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T} kt} dt \quad \text{where } \omega_0 = \frac{2\pi}{T}.
\]
Continuous-Time Fourier Series

Representing a periodic signal as a sum of harmonic sinusoids.

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

**analysis equation**

$$x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

**synthesis equation**

where $$\omega_0 = \frac{2\pi}{T}$$
Today: Discrete Time Fourier Series

Same high-level idea.

Express a periodic signal as a sum of harmonically related sinusoids

\[ x[n] = \sum_{k=0}^{\infty} (c_k \cos k\Omega_o n + d_k \sin k\Omega_o n) \]

where \( \Omega_o \) represents the fundamental frequency (radians/sample).

Basis functions:
Discrete-Time Fourier Series

Separating harmonic components relies on the same two key points.

1. Multiplying two DT harmonics produces a new DT harmonic with the same fundamental frequency:

\[ e^{jk\Omega_n} \times e^{jl\Omega_n} = e^{j(k+l)\Omega_n}. \]

2. The sum of a harmonic over any time interval with length equal to the period \( N \) is zero unless the harmonic is at DC:

\[
\sum_{n=n_0}^{n_0+N} e^{jk\Omega_n} \equiv \sum_{n=\langle N \rangle} e^{jk\Omega_n} = \begin{cases} 
N & \text{if } k = 0 \\
0 & \text{if } k \neq 0 
\end{cases} = N\delta[k].
\]

→ DT Fourier components are orthogonal.
Discrete-Time Fourier Series

Assume that $x[n]$ is periodic in $N$ and is composed of a weighted sum of harmonics of $\Omega_o = 2\pi/N$.

\[ x[n] = x[n + N] = \sum_k a_k e^{j\Omega_o kn} \]

Then “sift” out one component:

\[ \sum_{n=\langle N \rangle} x[n] e^{-j\Omega_o ln} = \sum_{n=\langle N \rangle} \sum_k a_k e^{j\Omega_o kn} e^{-j\Omega_o ln} \]

\[ = \sum_k a_k \sum_{n=\langle N \rangle} e^{j\Omega_o (k-l)n} \]

\[ = \sum_k a_k N \delta[k - l] = a_l N \]

Solving for $a_l$ provides an explicit formula for the coefficients:

\[ a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\Omega_o kn} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi}{N} kn} \]

where $\Omega_o = \frac{2\pi}{N}$. 
Discrete-Time Fourier Series

Representing a DT periodic signal as a sum of harmonic sinusoids.

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j k \Omega_0 n}$$

**analysis equation**

$$x[n] = x[n + N] = \sum_{k} a_k e^{j k \Omega_0 n}$$

**synthesis equation**

where \( \Omega_0 = \frac{2 \pi}{N} \)
Discrete-Time Fourier Series

Representing a DT periodic signal as a sum of harmonic sinusoids.

\[ a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j k \Omega_o n} \]  
**analysis equation**

\[ x[n] = x[n + N] = \sum_k a_k e^{j k \Omega_o n} \]  
**synthesis equation**

where \( \Omega_o = \frac{2\pi}{N} \)

DT Fourier series are similar to CT Fourier series ... but DT sinusoids **differ** from CT sinusoids in important ways!

How do CT/DT differences affect Fourier series?
What is the fundamental (shortest) period of each of the following signals?

1. $x_1[n] = \cos \frac{\pi n}{12}$

2. $x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15}$

3. $x_3[n] = \cos n + \cos 2n + \cos 3n$
Check Yourself

\[ x_1[n] = \cos \frac{\pi n}{12} = \cos \left( \frac{\pi n}{12} + 2\pi \right) = \cos \frac{\pi (n + 24)}{12} = x_1[n + 24] \]

\[ x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15} = \cos \left( \frac{\pi n}{12} + 10\pi \right) + 3 \cos \left( \frac{\pi n}{15} + 8\pi \right) \]

\[ = \cos \frac{\pi (n + 120)}{12} + 3 \cos \frac{\pi (n + 120)}{15} = x_2[n + 120] \]

\[ x_3[n] = \cos n + \cos 2n + \cos 3n \text{ is not periodic.} \]
Check Yourself

What is the fundamental (shortest) period of each of the following signals?

1. $x_1[n] = \cos \frac{\pi n}{12}$  \quad 24

2. $x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15}$  \quad 120

3. $x_3[n] = \cos n + \cos 2n + \cos 3n$  \quad \infty
Check Yourself

What is the fundamental (shortest) period of each of the following signals?

1. $x_1[n] = \cos \frac{\pi n}{12}$  \hspace{1cm} 24

2. $x_2[n] = \cos \frac{\pi n}{12} + 3 \cos \frac{\pi n}{15}$  \hspace{1cm} 120

3. $x_3[n] = \cos n + \cos 2n + \cos 3n$  \hspace{1cm} \infty

The period of a periodic DT signal must be an integer.

Therefore the fundamental frequency of a periodic DT signal must be an integer submultiple of $2\pi$.

No such constraints on fundamental frequencies in CT.
Discrete-Time Sinusoids

The frequencies of discrete-time sinusoids **alias**.

\[ \Omega = 0.25 \]

\[ x[n] = \cos(0.25n) \]

\[ n \]
Discrete-Time Sinusoids

The frequencies of discrete-time sinusoids alias.

\[ \Omega = 0.5 \]

\[ x[n] = \cos(0.5n) \]
Discrete-Time Sinusoids

The frequencies of discrete-time sinusoids alias.

\[ \Omega = 1 \]

\[ x[n] = \cos(n) \]
Discrete-Time Sinusoids

The frequencies of discrete-time sinusoids alias.

\[ \Omega = 2 \]

\[ x[n] = \cos(2n) \]
The frequencies of discrete-time sinusoids alias.

\[ \Omega = 3 \]

\[ x[n] = \cos(3n) \]
Discrete-Time Sinusoids

The frequencies of discrete-time sinusoids alias.

\[ \Omega = 4 \]

\[ x[n] = \cos(4n) = \cos(2\pi - 4n) \approx \cos(2.283n) \]
Discrete-Time Sinusoids

The frequencies of discrete-time sinusoids alias.

\[ \Omega = 5 \]

\[ x[n] = \cos(5n) = \cos(2\pi - 5n) \approx \cos(1.283n) \]
Discrete-Time Sinusoids

The frequencies of discrete-time sinusoids alias.

\( \Omega = 6 \)

\[ x[n] = \cos(6n) = \cos(2\pi - 6n) \approx \cos(0.283n) \]

There are multiple "frequencies" associated with every DT sinusoid. The frequencies of discrete-time sinusoids alias.
Discrete-Time Sinusoids

If \( \Omega_o \) is a submultiple of \( 2\pi \), and the harmonic frequencies alias, then there are (only) \( N \) distinct complex exponentials with period \( N \). (There were an infinite number in CT!)

If \( y[n] = e^{j\Omega n} \) is periodic in \( N \) then

\[
y[n] = e^{j\Omega n} = y[n + N] = e^{j\Omega(n+N)} = e^{j\Omega n} e^{j\Omega N}
\]

and \( e^{j\Omega N} \) must be 1 \( \rightarrow \) \( e^{j\Omega} \) must be one of the \( N^{th} \) roots of 1.

Example: \( N = 8 \)
Discrete-Time Sinusoids

There are $N$ distinct complex exponentials with period $N$.

If a DT signal is periodic with period $N$, then its Fourier series contains just $N$ terms.

Example: periodic in $N=3$

3 samples repeated in time

Example: periodic in $N=4$

4 samples repeated in time
Discrete-Time Fourier Series

DT Fourier series comprise a weighted sum of just $N$ harmonics.

$$x[n] = x[n + N] = \sum_{k=N}^{k=N} a_k e^{j\Omega_0 kn}$$
Discrete-Time Fourier Series

DT Fourier series comprise a weighted sum of just \( N \) harmonics.

\[
x[n] = x[n + N] = \sum_{k=\langle N \rangle} a_k e^{j\Omega_0 kn}
\]

Then “sift” out one component:

\[
\sum_{n=\langle N \rangle} x[n] e^{-j\Omega_0 ln} = \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{j\Omega_0 kn} e^{-j\Omega_0 ln}
\]

\[
= \sum_{k=\langle N \rangle} a_k \sum_{n=\langle N \rangle} e^{j\Omega_0 (k-l)n}
\]

\[
= \sum_{k=\langle N \rangle} a_k N \delta[k-l] = a_l N
\]

Solving for \( a_l \) provides an explicit formula for the coefficients:

\[
a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\Omega_0 kn} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi}{N} kn} \quad \text{where} \quad \Omega_0 = \frac{2\pi}{N}.
\]

Both \( x[n] \) and \( e^{-j\frac{2\pi}{N} kn} \) are periodic in \( N \), \( \therefore a_k \) is periodic in \( N \).
A periodic DT signal with \( N \) samples produces a periodic sequence of \( N \) Fourier series coefficients.

\[
a_k = a_{k+N} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]e^{-jk\Omega_o n}
\]

\[
x[n] = x[n+N] = \sum_{k=\langle N \rangle} a_ke^{jk\Omega_o n}
\]

where \( \Omega_o = \frac{2\pi}{N} \)
Fourier Series Summary

CT and DT Fourier Series are similar, but DT Fourier Series have just $N$ coefficients while CT Fourier Series have an infinite number.

Continuous-Time Fourier Series

$$a_k = \frac{1}{T} \int_T x(t)e^{-jk\omega_0 t} dt$$

analysis equation

$$x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

synthesis equation

where $\omega_0 = \frac{2\pi}{T}$

Discrete-Time Fourier Series

$$a_k = a_{k+N} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]e^{-jk\Omega_0 n}$$

analysis equation

$$x[n] = x[n + N] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n}$$

synthesis equation

where $\Omega_0 = \frac{2\pi}{N}$
Properties of Discrete-Time Fourier Series

Operations on the time representation of a signal can often be interpreted as equivalent operations on the series coefficients.

Example: Fourier series of a linear combination of signals

Proof: Let

\[ x[n] = ax_1[n] + bx_2[n] \quad \text{where} \quad x_1[n] = x_1[n+N] \quad \text{and} \quad x_2[n] = x_2[n+N] \]

then the Fourier series coefficients for \( x[n] \) are given by

\[
X[k] = \frac{1}{N} \sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=<N>} (ax_1[n] + bx_2[n])e^{-jk\frac{2\pi}{N}n} \\
= a \frac{1}{N} \sum_{n=<N>} x_1[n]e^{-jk\frac{2\pi}{N}n} + b \frac{1}{N} \sum_{n=<N>} x_2[n]e^{-jk\frac{2\pi}{N}n} \\
= aX_1[k] + bX_2[k]
\]

where \( X_1[k] \) and \( X_2[k] \) are Fourier series coefficients for \( x_1 \) and \( x_2 \).
Properties of Discrete-Time Fourier Series

Operations on the time representation of a signal can often be interpreted as equivalent operations on the series coefficients.

Example: Shifting time changes the phases of Fourier components.

Proof: Let
\[ y[n] = x[n - n_0] \quad \text{where} \quad x[n] = x[n + N] \]

If
\[ X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk \frac{2\pi}{N} n} \]

then
\[ Y[k] = \frac{1}{N} \sum_{n=\langle N \rangle} y[n] e^{-jk \frac{2\pi}{N} n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n - n_0] e^{-jk \frac{2\pi}{N} n} \]
\[ = \frac{1}{N} \sum_{m=\langle N \rangle} x[m] e^{-jk \frac{2\pi}{N} (m+n_0)} \quad \text{where} \quad m = n - n_0 \]
\[ = e^{-jk \frac{2\pi}{N} n_0} \frac{1}{N} \sum_{m=\langle N \rangle} x[m] e^{-jk \frac{2\pi}{N} m} = e^{-jk \frac{2\pi}{N} n_0} X[k] \]
Properties of Discrete-Time Fourier Series

Operations on the time representation of a signal can often be interpreted as equivalent operations on the series coefficients.

Example: Flipping time flips frequency.

Proof: Let

\[ y[n] = x[-n] \quad \text{where} \quad x[n] = x[n + N] \]

If

\[ X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi}{N} n} \]

then

\[ Y[k] = \frac{1}{N} \sum_{n=\langle N \rangle} y[n] e^{-j \frac{2\pi}{N} n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[-n] e^{-j \frac{2\pi}{N} n} \]

\[ = \frac{1}{N} \sum_{m=\langle N \rangle} x[m] e^{j \frac{2\pi}{N} m} \quad \text{where} \quad m = -n \]

\[ = X[-k] \]
Properties of Discrete-Time Fourier Series

Operations on the time representation of a signal can often be interpreted as equivalent operations on the series coefficients.

Example: If \( x[n] \) is a real-valued sequence, then \( X[-k] = X[k]^* \).

\[
X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j k \frac{2\pi}{N} n}
\]

\[
X[-k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{j k \frac{2\pi}{N} n} = X[k]^*
\]
Properties of Discrete-Time Fourier Series

Operations on the time representation of a signal can often be interpreted as equivalent operations on the series coefficients.

Example: Even and Odd Decomposition

Find the Fourier series for the even and odd parts of a signal.

\[
x[n] \, \overset{\text{FS}}{\leftrightarrow} \, X[k] \\
x[-n] \, \overset{\text{FS}}{\leftrightarrow} \, X[-k]
\]

\[
\text{Even}(x[n]) = \frac{1}{2}(x[n] + x[-n]) \, \overset{\text{FS}}{\leftrightarrow} \, \frac{1}{2}(X[k] + X[-k])
\]

If \( x[n] \) is real, then \( X[-k] = X[k]^* \).

\[
\text{Even}(x[n]) = \frac{1}{2}(x[n] + x[-n]) \, \overset{\text{FS}}{\leftrightarrow} \, \frac{1}{2}(X[k] + X[k]^*) = \text{Re}(X[k])
\]

\[
\rightarrow \quad \text{Even}(x[n]) \, \overset{\text{FS}}{\leftrightarrow} \, \text{Re}(X[k])
\]

Similarly

\[
\rightarrow \quad \text{Odd}(x[n]) \, \overset{\text{FS}}{\leftrightarrow} \, j\text{Im}(X[k])
\]
Summary

Discrete-Time Fourier Series

- orthogonality of harmonically related DT sinusoids
- DT Fourier series relations
- differences between CT and DT Fourier series
- properties of DT Fourier series

Today’s Lab:

Use DT Fourier Series to Understand an Auditory Perception.

For today only: Students who normally have lab in 1-150 should stay here instead.