Continuous-Time Fourier Series

- orthogonality of basis functions
- Fourier series analysis
- Fourier series synthesis
- convergence and Gibbs phenomenon
- properties of Fourier series
Fourier Representations of Signals

Signals are functions that are used to convey information.

Example: a musical sound can be represented as a function of time.

![Sound Pressure vs Time Graph](https://via.placeholder.com/150)

Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

Today’s goal is to develop methods to represent signals as sums of sinusoids.
Fourier Representations of Signals

The dominant sinusoidal components of many musical sounds are harmonically related – i.e., frequencies are integer multiples of a fundamental.
Harmonic representations can meaningfully characterize many musical sounds.

(from http://theremin.music.uiowa.edu)
Fourier Representations of Signals

**Fourier series** are sums of harmonically related sinusoids.

\[ x(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_0 t + d_k \sin k\omega_0 t) \]

where \( \omega_0 \) represents the fundamental frequency. Basis functions:

Q1: Under what conditions can we write \( x(t) \) as a Fourier series?

Q2: How do we find the coefficients \( c_k \) and \( d_k \).
Fourier Representations of Signals

What signals can be represented by sums of harmonic components?

Only periodic signals: all harmonics of $\omega_o$ are periodic in $T = \frac{2\pi}{\omega_o}$. 
Definition: a signal $x(t)$ is periodic in $T$ if

$$x(t) = x(t + T)$$

for all $t$.

Note: if a signal is periodic in $T$ it is also periodic in $2T$, $3T$, ...

The smallest positive number $T_o$ for which $x(t) = x(t + T_o)$ for all $t$ is sometimes called the **fundamental period**.

If a signal does not satisfy $x(t) = x(t + T)$ for any value of $T$, then the signal is **aperiodic**.
Assume that \( x(t) \) is periodic in \( T \) and is composed of a weighted sum of harmonics of \( \omega_0 = \frac{2\pi}{T} \).

\[
x(t) = x(t + T) = \sum_{k=0}^{\infty} \left( c_k \cos k\omega_0 t + d_k \sin k\omega_0 t \right)
\]

How do we find the weights?
Complex Exponentials

We can simplify the notation (and resulting work) by using complex exponentials to represent the trigonometric functions.

Start with the previous expression:

\[ x(t) = x(t + T) = \sum_{k=0}^{\infty} (c_k \cos k\omega t + d_k \sin k\omega t) . \]

Substitute

\[ \cos k\omega t = \text{Re} \left( e^{jk\omega t} \right) = \frac{1}{2} \left( (e^{jk\omega t}) + (e^{jk\omega t})^* \right) = \frac{1}{2} \left( e^{jk\omega t} + e^{-jk\omega t} \right) \]
\[ \sin k\omega t = \text{Im} \left( e^{jk\omega t} \right) = \frac{1}{2j} \left( (e^{jk\omega t}) - (e^{jk\omega t})^* \right) = -\frac{j}{2} \left( e^{jk\omega t} - e^{-jk\omega t} \right) \]

to get

\[ x(t) = \sum_{k=0}^{\infty} \left( \frac{c_k - jd_k}{2} e^{jk\omega t} + \frac{c_k + jd_k}{2} e^{-jk\omega t} \right) \]
\[ = \sum_{k=0}^{\infty} \left( b_k e^{jk\omega t} + b_k^* e^{-jk\omega t} \right) \text{ where } b_k = \frac{c_k - jd_k}{2} . \]
Complex Exponentials

We can simplify even further by introducing “negative frequencies.”

\[ x(t) = \sum_{k=0}^{\infty} \left( b_k e^{jk\omega t} + b^*_k e^{-jk\omega t} \right) \]

\[ = \sum_{k=0}^{\infty} b_k e^{jk\omega t} + \sum_{k=0}^{\infty} b^*_k e^{-jk\omega t} \]

\[ = \sum_{k=0}^{\infty} b_k e^{jk\omega t} + \sum_{k=-\infty}^{0} b^*_k e^{jk\omega t} \]

\[ = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} \quad \text{where} \quad a_k = \begin{cases} 
  b_k & \text{if } k > 0 \\
  2b_0 & \text{if } k = 0 \\
  b^*_{-k} & \text{if } k < 0 
\end{cases} \]

A single complex exponential series replaces the cos and sin series.

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} \]

How do we find the \( a_k \) coefficients?
## Separating Harmonic Components

Two key observations:

1. Multiplying two harmonics produces a new harmonic with the same fundamental frequency:

\[ e^{j k \omega_0 t} \times e^{j l \omega_0 t} = e^{j (k+l) \omega_0 t}. \]

2. The integral of a harmonic over any time interval with length equal to the period \( T \) is zero unless the harmonic is at DC:

\[
\int_{t_0}^{t_0+T} e^{j k \omega_0 t} dt = \int_{T} e^{j k \omega_0 t} dt = \begin{cases} T & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} = T \delta[k].
\]

→ Fourier components are **orthogonal**.


Separating Harmonic Components

Assume that \( x(t) \) is periodic in \( T \) and is composed of a weighted sum of harmonics of \( \omega_o = 2\pi/T \).

\[
x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}
\]

Then “sift” out one component:

\[
\int_T x(t) e^{-j\omega_0 lt} dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} e^{-j\omega_0 lt} dt
\]

\[
= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_0(k-l)t} dt
\]

\[
= \sum_{k=-\infty}^{\infty} a_k T \delta[k-l] = a_l T
\]

Solving for \( a_l \) provides an explicit formula for the coefficients:

\[
a_k = \frac{1}{T} \int_T x(t) e^{-j\omega_0 kt} dt = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T} kt} dt \quad \text{where} \quad \omega_o = \frac{2\pi}{T}.
\]
Fourier Series

Determining harmonic components of a periodic signal.

\[ a_k = \frac{1}{T} \int_{T} x(t) e^{-j \frac{2\pi}{T} kt} dt \]  
(“analysis” equation)

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j \frac{2\pi}{T} kt} \]  
(“synthesis” equation)
How many of the following pairs of functions are orthogonal (⊥) in $T = 3$?

1. $\cos 2\pi t \perp \sin 2\pi t$?
2. $\cos 2\pi t \perp \cos 4\pi t$?
3. $\cos 2\pi t \perp \sin \pi t$?
4. $\cos 2\pi t \perp e^{j2\pi t}$?
How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \sin 2\pi t$?
Check Yourself

How many of the following are orthogonal (⊥) in $T = 3$?

$\cos 2\pi t \perp \sin 2\pi t$?

$\cos 2\pi t \perp \sin 2\pi t$?

$\cos 2\pi t \cdot \sin 2\pi t = \frac{1}{2} \sin 4\pi t$

$\int_{0}^{3} dt = 0 \therefore \text{YES}$
Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \cos 4\pi t$ ?
Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \cos 4\pi t$?

$\cos 2\pi t \cos 4\pi t = \frac{1}{2} \cos 6\pi t + \frac{1}{2} \cos 2\pi t$

$\int_0^3 dt = 0$ therefore YES
Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \sin \pi t$?
Check Yourself

How many of the following are orthogonal \((\perp)\) in \(T = 3\)?

\[ \cos 2\pi t \perp \sin \pi t? \]

\[ \int_{0}^{3} dt \neq 0 \text{ therefore } \text{NO} \]
Check Yourself

How many of the following are orthogonal (⊥) in $T = 3$?

$\cos 2\pi t \perp e^{j2\pi t}$?
Check Yourself

How many of the following are orthogonal (\( \perp \)) in \( T = 3 \)?

\[
\cos 2\pi t \perp e^{j2\pi t} ?
\]

\[
e^{2\pi t} = \cos 2\pi t + j \sin 2\pi t
\]

\[
\cos 2\pi t \perp \sin 2\pi t \text{ but not } \cos 2\pi t
\]

Therefore \textbf{NO}
Check Yourself

How many of the following pairs of functions are orthogonal (⊥) in $T = 3$? 2

1. $\cos 2\pi t \perp \sin 2\pi t$? \checkmark
2. $\cos 2\pi t \perp \cos 4\pi t$? \checkmark
3. $\cos 2\pi t \perp \sin \pi t$? \xmark
4. $\cos 2\pi t \perp e^{j2\pi t}$? \xmark
Check Yourself

Let $a_k$ represent the Fourier series coefficients of the following square wave.

$x(t) = \begin{cases} 
1/2 & 0 < t < 1/2 \\
-1/2 & 1/2 < t < 1 \\
0 & \text{otherwise}
\end{cases}$

How many of the following statements are true?

1. $a_k = 0$ if $k$ is even
2. $a_k$ is real-valued
3. $|a_k|$ decreases with $k^2$
4. there are an infinite number of non-zero $a_k$
5. all of the above
Let $a_k$ represent the Fourier series coefficients of the following square wave.

$$
\begin{align*}
    a_k &= \int_{T} x(t) e^{-j \frac{2\pi kt}{T}} dt = -\frac{1}{2} \int_{-\frac{1}{2}}^{0} e^{-j 2\pi kt} dt + \frac{1}{2} \int_{0}^{\frac{1}{2}} e^{-j 2\pi kt} dt \\
    &= - \frac{e^{-j 2\pi k t}}{-j 4\pi k} \bigg|_{-\frac{1}{2}}^{0} + \frac{e^{-j 2\pi k t}}{-j 4\pi k} \bigg|_{0}^{\frac{1}{2}} = \frac{1}{j 4\pi k} \left(2 - e^{j \pi k} - e^{-j \pi k}\right) \\
    &= \begin{cases} 
    \frac{1}{j \pi k} ; & \text{if } k \text{ is odd} \\
    0 ; & \text{otherwise}
    \end{cases}
\end{align*}
$$
Check Yourself

Let \( a_k \) represent the Fourier series coefficients of the following square wave.

\[
a_k = \begin{cases} 
\frac{1}{j\pi k} & \text{if } k \text{ is odd} \\
0 & \text{otherwise}
\end{cases}
\]

How many of the following statements are true?

1. \( a_k = 0 \) if \( k \) is even  \( \checkmark \)
2. \( a_k \) is real-valued  \( \times \)
3. \( |a_k| \) decreases with \( k^2 \)  \( \times \)
4. there are an infinite number of non-zero \( a_k \)  \( \checkmark \)
5. all of the above  \( \times \)
Check Yourself

Let \( a_k \) represent the Fourier series coefficients of the following square wave.

\[
x(t) = \begin{cases} 
1/2 & \text{if } 0 \leq t < 1/2 \\
1 & \text{if } 1/2 \leq t < 1 \\
-1/2 & \text{if } 1 \leq t < 1
\end{cases}
\]

How many of the following statements are true? 2

1. \( a_k = 0 \) if \( k \) is even \( \sqrt{ } \)
2. \( a_k \) is real-valued \( \times \)
3. \( |a_k| \) decreases with \( k^2 \) \( \times \)
4. there are an infinite number of non-zero \( a_k \) \( \sqrt{ } \)
5. all of the above \( \times \)
Fourier Series Properties

Operations on the time representation of a signal can often be interpreted as equivalent operations on the series coefficients.

Example: Fourier series of a sum is the sum of the Fourier series.

Proof: Let

\[ x(t) = x_1(t) + x_2(t) \quad \text{where} \quad x_1(t) = x_1(t+T) \quad \text{and} \quad x_2(t) = x_2(t+T) \]

then the Fourier series coefficients for \( x(t) \) are given by

\[
a_k = \frac{1}{T} \int_T x(t) e^{-jk\frac{2\pi}{T}t} dt = \frac{1}{T} \int_T (x_1(t) + x_2(t)) e^{-jk\frac{2\pi}{T}t} dt
\]

\[
= \frac{1}{T} \int_T x_1(t) e^{-jk\frac{2\pi}{T}t} dt + \frac{1}{T} \int_T x_2(t) e^{-jk\frac{2\pi}{T}t} dt
\]

\[
= b_k + c_k
\]

where \( b_k \) and \( c_k \) are the Fourier series coefficients for \( x_1(t) \) and \( x_2(t) \).
Fourier Series Properties

Operations on the time representation of a signal can often be interpreted as equivalent operations on the series coefficients.

Example: Shifting time changes the phases of Fourier components.

Proof: Let
\[ y(t) = x(t - t_0) \quad \text{where} \quad x(t) = x(t + T) \]

If
\[ a_k = \frac{1}{T} \int_T x(t) e^{-j k \frac{2\pi}{T} t} \, dt \]
then
\[ b_k = \frac{1}{T} \int_T y(t) e^{-j k \frac{2\pi}{T} t} \, dt = \frac{1}{T} \int_T x(t - t_0) e^{-j k \frac{2\pi}{T} t} \, dt \]
\[ = \frac{1}{T} \int_T x(\tau) e^{-j k \frac{2\pi}{T} (\tau + t_0)} \, d\tau \quad \text{where} \quad \tau = t - t_0 \]
\[ = e^{-j k \frac{2\pi}{T} t_0} \frac{1}{T} \int_T x(\tau) e^{-j k \frac{2\pi}{T} \tau} \, d\tau = e^{-j k \frac{2\pi}{T} t_0} a_k \]
Fourier Series Properties

Operations on the time representation of a signal can often be interpreted as equivalent operations on the series coefficients.

Example: If a signal is differentiated in time, its Fourier coefficients are multiplied by $j\frac{2\pi}{T}k$.

Proof: Let

$$x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

then

$$\dot{x}(t) = \dot{x}(t + T) = \sum_{k=-\infty}^{\infty} \left( j\frac{2\pi}{T}k a_k \right) e^{j\frac{2\pi}{T}kt}$$

$$x(t) \overset{\text{CTFS}}{\leftrightarrow} a_k$$

$$\dot{x}(t) \overset{\text{CTFS}}{\leftrightarrow} j\frac{2\pi}{T}k a_k$$
Check Yourself

Let $b_k$ represent the Fourier series coefficients of the following triangle wave.

$y(t)$

1/8

1/8

0 1

How many of the following statements are true?

1. $b_k = 0$ if $k$ is even
2. $b_k$ is real-valued
3. $|b_k|$ decreases with $k^2$
4. there are an infinite number of non-zero $b_k$
5. all of the above
Check Yourself

The triangle waveform is the integral of the square wave.

\[
x(t) = 1/2 - 1/2^t \quad 0 \leq t \leq 1
\]

\[
y(t) = 1/8 - 1/8^t \quad 0 \leq t \leq 1
\]

Therefore the Fourier coefficients of the triangle waveform are \(1/j2\pi k\) times those of the square wave.

\[
b_k = \frac{1}{jk\pi} \times \frac{1}{j2\pi k} = \frac{-1}{2k^2\pi^2} \quad k \text{ odd}
\]
Check Yourself

Let $b_k$ represent the Fourier series coefficients of the following triangle wave.

$$b_k = \frac{-1}{2k^2\pi^2}; \ k \text{ odd}$$

How many of the following statements are true?

1. $b_k = 0$ if $k$ is even $\checkmark$
2. $b_k$ is real-valued $\checkmark$
3. $|b_k|$ decreases with $k^2$ $\checkmark$
4. there are an infinite number of non-zero $b_k$ $\checkmark$
5. all of the above $\checkmark$
Check Yourself

Let $b_k$ represent the Fourier series coefficients of the following triangle wave.

How many of the following statements are true? 5

1. $b_k = 0$ if $k$ is even  √
2. $b_k$ is real-valued  √
3. $|b_k|$ decreases with $k^2$  √
4. there are an infinite number of non-zero $b_k$  √
5. all of the above  √
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[
\sum_{k = -\infty}^{\infty} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}
\]

\[y(t)\]

\[\frac{1}{8}\]

\[0\]

\[-\frac{1}{8}\]

\[t\]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[ y(t) = \sum_{k=-1}^{k \text{ odd}} \frac{1}{2k^2 \pi^2} e^{j2\pi kt} \frac{-1}{2} \]

\[ y(t) = \frac{1}{8} - \frac{1}{8} \]

\[ 0 \quad 1 \]

\[ t \]

\[ y(t) \]

\[ \frac{1}{8} \]

\[ -\frac{1}{8} \]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[
\sum_{k=\text{odd}}^{3} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}
\]

\[
y(t) = \frac{1}{8} - \frac{1}{8} \quad 0 \leq t \leq 1
\]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[
\sum_{k = -5}^{5} \frac{-1}{2k^2 \pi^2} e^{j2\pi kt}
\]

\[y(t)\]

0 1

-1/8 1/8
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[
\sum_{k=\text{odd}}^{7} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}
\]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[ y(t) = \sum_{k=-9}^{9} \frac{-1}{2k^2 \pi^2} e^{j2\pi kt} \]

\( k \) odd
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[
\sum_{k \text{ odd}}^{19} -\frac{1}{2k^2\pi^2} e^{j2\pi kt}
\]

\[y(t)\]

\[1/8\]

\[0\]

\[-1/8\]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[
\sum_{k=\text{odd}}^{29} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}
\]

\[y(t)\]

\[1/8\]

\[0\]

\[-1/8\]

\[t\]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

\[ y(t) = \sum_{k=\text{odd}}^{39} \left( -\frac{1}{2k^2\pi^2} e^{j2\pi kt} \right) \]

Fourier series representations of functions with discontinuous slopes converge toward functions with discontinuous slopes.
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

\[ x(t) = \sum_{k=\text{odd}}^{\infty} \frac{1}{jk\pi} e^{j2\pi kt} \]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$x(t) = \sum_{k=-3}^{3} \frac{1}{jk\pi} e^{j2\pi kt}$$

$\frac{1}{2}$ $\frac{-1}{2}$ $1$ $0$ $t$
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

\[ x(t) = \sum_{k = -5}^{5} \frac{1}{jk\pi} e^{j2\pi kt} \]

where \( k \) is odd.
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

\[
\sum_{k=-7}^{7} \frac{1}{jk\pi} e^{j2\pi kt}
\]

\(x(t)\)

\(t\)
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

\[ x(t) = \sum_{k=\text{odd}}^{9} \frac{1}{jk\pi} e^{j2\pi kt} \]

\[ x(t) = x(t) \]

\[ 1/2 \]

\[ -1/2 \]

\[ 0 \]

\[ 1 \]

\[ t \]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

\[ x(t) = \sum_{k=-19}^{19} \frac{1}{jk\pi} e^{j2\pi kt} \]

\[ k \text{ odd} \]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

\[ x(t) = \frac{1}{2} + \sum_{k=-29}^{29} \sum_{k \text{ odd}} \frac{1}{jk\pi} e^{j2\pi kt} \]
Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

\[
\sum_{k = -39}^{39} \frac{1}{jk\pi} e^{j2\pi kt}
\]

\[x(t)\]

\[\frac{1}{2}\quad 0 \quad 1\]

\[-\frac{1}{2}\]
Partial sums of Fourier series of discontinuous functions “ring” near discontinuities: Gibb’s phenomenon.

This ringing results because the magnitude of the Fourier coefficients is only decreasing as $\frac{1}{k}$ (while they decreased as $\frac{1}{k^2}$ for the triangle).

You can decrease (and even eliminate the ringing) by decreasing the magnitudes of the Fourier coefficients at higher frequencies.