6.003: Signal Processing

**Building Block Signals**
- representing a signal as a sum of simpler signals
- representing a sample as a sum of samples from simpler signals

**Reminders From Last Time**
Piloting a new version of 6.003 focused on **Signal Processing**.
Combines **theory**
- analysis and synthesis of signals,
- time and frequency domains,
- convolution and deconvolution,
- filtering and noise reduction,
with authentic, real-world **applications** in
- music,
- imaging,
- video.

Authentic connections to real-world applications are key to developing a **deep and useful** understanding of the content.

**Importance of Computation**
Our goal is to develop theories that help solve real-world problems.

```
Model          → analyze (math, computation) → Result
make model     → interpret results
World          → New Understanding
```

**Classical analyses** use a variety of maths, especially **calculus**.
We will also use **computation** which is applicable in many real-world problems that are difficult or impossible to solve analytically → **strengthens ties to the real world**.

**Design as the Reverse of Analysis**
We are interested not only in analyzing the behaviors of pre-existing systems, but also in designing new systems.

```
System          → analysis → Behavior
design
```

Analysis and Design are **different** and **complementary** activities. **Analysis** starts with a precise statement of the problem and proceeds to a precise statement of a result. → **conventional problem sets**

**Design** is more open-ended, with multiple possible solutions that differ along idiosyncratic dimensions that were not even part of the original problem statement. → **Engineering Design Problems**

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1 adapted from R. David Middlebrook
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**Weekly Activities**

**Lecture**: Tuesdays and Thursdays 2-3pm in 1-190

**Labs**: Tues. and Thur. 3-5pm in 4-145, 1-150, 4-153, 1-190

**Homework** – issued Tuesdays, due following Tuesday at noon
- **Drills**: facts, definitions, and simple concepts
  - online with immediate feedback (not graded)
  - intended as practice and self-assessment
- **Problems** – intended to improve problem solving skills
  - **Exam-Type Problems**: proveably correct solutions
  - **Engineering Design Problems**: real-world applications (more open ended)

**Piazza** – simple answers to simple questions

**Office Hours** – better for deeper more conceptual questions
- Thursdays 7pm-9pm 38-530
- Sundays 11am-5pm 38-530

**Meetings** with any staff member by appointment.

**Advisory Group**
Weekly meetings with **class representatives** start tomorrow:
- help staff understand student perspective
- learn about teaching

We will meet on Wednesdays at 3pm.
Interested? ... Send email to **freeman@mit.edu**
**Sounds as Signals**

Signals are functions that are used to convey information.

Example: a musical sound can be represented as a function of time.

Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

**Expressing Signals as Sums of Simpler Signals**

As a simpler example, consider expansions based on derivatives.

**Maclaurin expansion:**

\[ f(t) = f(0) + \frac{f'(0)}{1!}t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \ldots \]

The function of time \( f(t) \) is represented by its derivatives at \( t = 0 \).

**Taylor expansion** about \( t = a \):

\[ f(t) = f(a) + \frac{f'(a)}{1!}(t-a) + \frac{f''(a)}{2!}(t-a)^2 + \frac{f'''(a)}{3!}(t-a)^3 + \ldots \]

The function of time \( f(t) \) is represented by its derivatives at \( t = a \).

Here, the **basis functions** are polynomials, and the **coefficients** are proportional to successive derivatives at \( t = 0 \) or \( t = a \).

**Series Representations of Signals**

Maclaurin series

\[ f(t) = f(0) + f'(0)\frac{t}{1!} + f''(0)\frac{t^2}{2!} + f'''(0)\frac{t^3}{3!} + \ldots \]

Basis functions:

<table>
<thead>
<tr>
<th>( t/0! )</th>
<th>( t/1! )</th>
<th>( t^2/2! )</th>
<th>( t^3/3! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>( t )</td>
<td>( t^2 )</td>
<td>( t^3 )</td>
</tr>
</tbody>
</table>

Notice that even powers of \( t \) are **even functions** of time, and odd powers of \( t \) are **odd functions** of time.

The expansion for \( t > 0 \) implicitly determines the function for \( t < 0 \).

**Right-Sided Basis Functions**

To model a signal that **starts** at a particular time (say \( t = 0 \)), we can use basis functions that start at \( t = 0 \).

\[ f(t) = f(0)u_0(t) + f'(0)u_2(t) + f''(0)u_2(t) + f'''(0)u_3(t) + \ldots \]

Basis functions:

\[ u_0(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \geq 1 \end{cases} \]

\[ u_2(t) = \frac{t}{1} u_0(t) \]

\[ u_3(t) = \frac{t^2}{1} u_0(t) \]

The first of these functions is the **unit step** \( u(t) = u_0(t) \). Subsequent functions are integrals of their predecessors \( u_{n-1}(t) = \int_0^t u_0(\tau)d\tau \).

**Visualizing Series**

Maclaurin expansion of the cosine function \( t > 0 \).

\[ f(t) = \cos(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2n!} \]

\[ f(t) = 1 + \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{720} + \frac{t^8}{30240} + \frac{t^{10}}{130767200} + \frac{t^{12}}{554214817120} \]

Both views are useful. For example,

- the peak sound pressure is more easily seen in \( f(t) \), while
- consonance is easier to see from the frequency components.
### Two Views

The signal \( \cos t \) can be represented as a function of time \( f(t) \) or as a sequence of coefficients \( F[k] \).

\[
f(t) = f(0) \frac{1}{0!} + f'(0) \frac{t}{1!} + f''(0) \frac{t^2}{2!} + f'''(0) \frac{t^3}{3!} + \cdots
\]

\[
= f(0) u_0(t) + f'(0) u_1(t) + f''(0) u_2(t) + f'''(0) u_3(t) + \cdots
\]

- \( f(0) = 1 \)
- \( f'(0) = 0 \)
- \( f''(0) = -1 \)
- \( f'''(0) = 0 \)
- \( f^{(4)}(0) = 1 \)
- \( f^{(5)}(0) = 0 \)
- \( f^{(6)}(0) = -1 \)

\[
F[k] = [1, 0, -1, 0, 1, 0, -1, 0, \ldots]
\]

### An Alternative View

The signal \( \sin t \) can be represented as a function of time \( g(t) \) or as a sequence of coefficients \( G[k] \).

\[
g(t) = g(0) \frac{1}{0!} + g'(0) \frac{t}{1!} + g''(0) \frac{t^2}{2!} + g'''(0) \frac{t^3}{3!} + \cdots
\]

\[
= g(0) u_0(t) + g'(0) u_1(t) + g''(0) u_2(t) + g'''(0) u_3(t) + \cdots
\]

- \( g(0) = 1 \)
- \( g'(0) = 0 \)
- \( g''(0) = 1 \)
- \( g'''(0) = 0 \)
- \( g^{(4)}(0) = 1 \)
- \( g^{(5)}(0) = 0 \)

\[
G[k] = \begin{cases} 0 & \text{if } k = 0 \\ F[k-1] & \text{otherwise} \end{cases}
\]

### Series Representations of Signals

This set of functions \( \{u_0(t), u_1(t), u_2(t), \ldots\} \) is closed under integration, i.e., if \( f(t) \) can be expressed as a sum of these functions, then the integral of \( f(t) \) can also.

**Example:**

\[
f(t) = \cos(t)u(t) = u(t) - \frac{t^2}{2!}u(t) + \frac{t^4}{4!}u(t) - \frac{t^6}{6!}u(t) + \cdots
\]

\[
= u_0(t) - u_2(t) + u_4(t) - u_6(t) + \cdots
\]

\[
g(t) = \int_{-\infty}^{t} f(r)dr = \sin(t)u(t) = tu(t) - \frac{t^3}{3!}u(t) + \frac{t^5}{5!}u(t) - \frac{t^7}{7!}u(t) + \cdots
\]

\[
= u_1(t) - u_3(t) + u_5(t) - u_7(t) + \cdots
\]

Since \( f(t) \) can be written as a sum of functions from the set, it follows that \( g(t) = \int_{-\infty}^{t} f(r)dr \) can also.

\( f(t) \) is represented by the sequence of coefficients \( [1, 0, -1, 0, 1, 0, -1, 0, \ldots] \)

\( g(t) \) is represented by the sequence of coefficients \( [0, 1, 0, -1, 0, 1, 0, -1, \ldots] \)

### Visualizing Series

**Maclaurin Expansion of the Sine Function (t > 0).**

\[
f(t) = \sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}
\]

### Summary: Series Representations of Signals

When we expand a function as a Maclaurin series, we are developing an alternative representation of that function.

Alternative representations can lead to new approaches/operations: e.g., integration in time \( \rightarrow \) shifting Maclaurin coefficients.

Alternative representations can even new insights (see today’s lab).
Pulses as Building Block Signals

We tend to think about the time representation of a 1D signal as the representation of a signal. But the time representation can just as easily be regarded as an expansion using a different set of basis functions. This is especially clear for discrete-time (DT) signals.

Pulses as Building Block Signals

The unit-sample signal \( \delta[n] \) is the simplest non-trivial DT signal. It has a single non-zero sample:
\[
\delta[n] = \begin{cases} 
1, & \text{if } n = 0; \\
0, & \text{otherwise}
\end{cases}
\]

This signal is useful for constructing more complex signals.

Unit-Impulse Signal

The unit-impulse function is represented by an arrow with the number 1, which represents its area or “weight.”

It has two seemingly contradictory properties:
- it is nonzero only at \( t = 0 \), and
- its definite integral \((-\infty, \infty)\) is one!

Both of these properties follow from thinking about \( \delta(t) \) as a limit:
\[
\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)
\]

The weights are the sample values: \( \cdots, x[-2], x[-1], x[0], x[1], x[2], \cdots \).

Unit-Impulse Signal

The impulse idea is analogous to the notion of a point mass. If the mass of a point were really a positive number, then the density would be infinite.
**Unit-Impulse Signal**

Impulses are useful for thinking about sampling and reconstruction. Sampling is straightforward:

\[ x[n] = x(nT) \]

Reconstruction is trickier. How can we reconstruct a continuous-time signal from its samples?

Exact reconstruction is not possible because sampling discards information.

However, we can approximate the CT signal with a sum of small increments (Riemann sum), and then represent each piece with an appropriately weighted impulse.

\[ y(t) = \sum_{n} x[n] W \delta(t - nT) \]

We will look at sampling more closely in the next lecture.

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**Representing a Sample as a Sum**

When we represent signals as sums of signals, we are implicitly representing samples as sums of samples.

Example. Determine the sum of an infinite geometric series.

\[
C = \sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + \cdots
\]

**Check Yourself**

Assume that we sample a CT signal \( x(t) \) once every \( T \) seconds

\[ x(t) \quad x[n] = x(nT) \]

and use the samples to make a signal \( y(t) \) to approximate \( x(t) \):

\[ y(t) = \sum_{n} x[n] W \delta(t - nT) \]

What should be the weight \( W \) of the impulses so that the reconstruction converges to \( x(t) \) as \( T \to 0 \).

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**Focus on Sinusoidal Basis Functions**

We can simplify the notation (and resulting work) by using complex exponentials to represent trigonometric functions:

\[ e^{j\theta} = \cos \theta + j \sin \theta \quad (\text{Euler's Equation}) \]

Now a single complex number captures both \( \cos \) and \( \sin \) dependence,

\[
x(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega t + d_k \sin k\omega t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}
\]

but we need to include both positive and negative values of \( k \) since

\[
\cos k\omega t = \frac{1}{2} e^{jk\omega t} + \frac{1}{2} e^{-jk\omega t} \quad \text{and} \quad \sin k\omega t = \frac{1}{j} e^{jk\omega t} - \frac{1}{j} e^{-jk\omega t}
\]

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**Note on Euler's Equation**

Euler’s equation extends the exponential function (with real domain) to a complex function of complex domain.

\[
e^\theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \frac{\theta^6}{6!} + \frac{\theta^7}{7!} + \cdots
\]

\[
e^{j\theta} = 1 + j\theta + \frac{j^2 \theta^2}{2!} + \frac{j^3 \theta^3}{3!} + \frac{j^4 \theta^4}{4!} + \frac{j^5 \theta^5}{5!} + \frac{j^6 \theta^6}{6!} + \frac{j^7 \theta^7}{7!} + \cdots
\]

\[
e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j \theta^3}{3!} - \frac{\theta^4}{4!} - \frac{j \theta^5}{5!} - \frac{\theta^6}{6!} - \frac{j \theta^7}{7!} - \cdots
\]

\[
e^{j\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{j \theta^3}{3!} - \frac{\theta^4}{4!} + \frac{j \theta^5}{5!} - \frac{\theta^6}{6!} + \frac{j \theta^7}{7!} - \cdots\right) + j \left(\theta - \frac{\theta^3}{3!} + \frac{j \theta^4}{4!} - \frac{\theta^5}{5!} + \frac{j \theta^6}{6!} - \frac{\theta^7}{7!} + \cdots\right)
\]

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

Richard Feynman called Euler’s equation “the most remarkable formula in mathematics.”
Note on Euler’s Equation

Euler’s equation can be interpreted as a relation between polar and cartesian coordinates of a unit vector at angle $\theta$.

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

If $\theta = \omega t$ then the angle $\theta$ increases at a constant rate (in radians/second or cycles/sec).

The real part is then $\cos \omega t$ and the imaginary part is $\sin \omega t$.

Rules for Combining Complex Numbers

Manipulating complex expressions is easier ...

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

\[ (a + jb) + (c + jd) = (a + c) + j(b + d) \]

\[ (a + jb) \times (c + jd) = (ac - bd) + j(ad + bc) \]

... but still takes practice.

Check Yourself

Complex numbers.

How many of the following are true?

A. \( \frac{1}{\cos \theta + j \sin \theta} = \cos \theta - j \sin \theta \)

B. \((\cos \theta + j \sin \theta)^n = \cos n\theta + j \sin n\theta \)

C. \( |2 + j2 + e^{j\pi/4}| = |2 + j2| + |e^{j\pi/4}| \)

D. \( \text{Im}(j^3) > \text{Re}(j^3) \)

E. \( \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{5} = \tan^{-1} 1 \)

0. 0 1. 1 2. 2 3. 3 4. 4 5. 5