Quiz #2

Name:
Kerberos Username:

Enter all answers in the boxes provided.
Other work on pages with QR codes may be considered when assigning partial credit.

You have two hours.
This quiz is closed book, but you may use two 8.5 × 11 sheets of paper (four sides).
No calculators, computers, cell phones, music players, or other electronic devices.
1. Shapes  [25 points]

Each of the images below was created by computing the inverse DFT on a 24-by-24 array of DFT coefficients, of which at most 5 were non-zero. In each of these images, black represents a value of 0, and white represents a value of 1. The origin of each image is in its center, \( n_x \) increases to the right, and \( n_y \) increases downward.

For each image, enter the locations \((k_x, k_y)\) of all non-zero values in the associated DFT. If the image could not have been made from an array of the form described above, enter None in the box instead.

A  B  C
D  E  F
G  H  I
J  K  L

Enter your answers on the facing page.
What are the locations of the nonzero values in the DFT associated with image A?

What are the locations of the nonzero values in the DFT associated with image B?

What are the locations of the nonzero values in the DFT associated with image C?

What are the locations of the nonzero values in the DFT associated with image D?

What are the locations of the nonzero values in the DFT associated with image E?

What are the locations of the nonzero values in the DFT associated with image F?

What are the locations of the nonzero values in the DFT associated with image G?

What are the locations of the nonzero values in the DFT associated with image H?

What are the locations of the nonzero values in the DFT associated with image I?

What are the locations of the nonzero values in the DFT associated with image J?

What are the locations of the nonzero values in the DFT associated with image K?

What are the locations of the nonzero values in the DFT associated with image L?
2. Convolution  [25 points]

Part 1. Consider a signal $x$, which is known to have the following values:

$$x[n] = \begin{cases} 
0 & \text{if } n < 0 \\
40 & \text{if } n = 0 \\
14 & \text{if } n = 1 \\
32 & \text{if } n = 2 \\
7 & \text{if } n = 3 
\end{cases}$$

All other values of $x$ are unknown.

When $x$ is the input to a linear, time-invariant (LTI) system whose unit sample response $f[n]$ is 0 for $n < 0$, its output is given by:

$$y[n] = \begin{cases} 
0 & \text{if } n < 0 \\
80 & \text{if } n = 0 \\
68 & \text{if } n = 1 \\
98 & \text{if } n = 2 \\
63 & \text{if } n = 3 
\end{cases}$$

All other values of $y$ are also unknown.

For what values of $n$ can $f[n]$ be unambiguously determined? Specify these values of $n$, as well as the associated values $f[n]$, in the box below.
Part 2. Let \( y_1[n] \) represent the result of convolving an input signal \( x[n] \) with the unit sample response \( g[n] \) of an LTI system (that is different from the one in part 1):

\[
y_1[n] = (x * g)[n]
\]

and let \( y_2[n] \) represent the result of convolving \( y_1[n] \) with the unit-sample response \( g[n] \) of a second, identical LTI system,

\[
y_2[n] = (y_1 * g)[n]
\]

as illustrated in the following figure:

We can think about this as convolving \( x \) with a different signal \( g_2 \):

\[
y_2[n] = (y_1 * g)[n] = (x * g_2)[n]
\]

When the input \( x[n] \) is equal to a unit-sample signal \( \delta[n] \), the output \( y_2[n] \) is

\[
y_2[n] = \delta[n] - 2\delta[n - 3] + \delta[n - 6]
\]

Determine the frequency response of \( g_2 \):

\[
G_2(\Omega) =
\]

Determine a closed-form expression (no infinite sums) for \( g[n] \):

\[
g[n] =
\]
Determine the frequency response $G_3(\Omega)$ of a new system that consists of three copies of the original system, such that

$$y_3[n] = (y_2 \ast g)[n] = (x \ast g_3)[n]$$

![System Diagram]

Enter a closed-form solution in the box below. You do not need to simplify your answer completely.

$$G_3(\Omega) =$$
Part 3. Consider an LTI system whose unit sample response is given by the following:

\[ h[n] = \delta[n] + \alpha\delta[n - m] + \alpha^2\delta[n - 2m] + \alpha^3\delta[n - 3m] + \ldots \]

where \( \alpha \) is a real-valued constant whose magnitude is less than 1, and \( m \) is a positive integer. Determine a closed-form expression for the frequency response of this system, in terms of \( \alpha \) and \( m \).

\[ H(\Omega) = \]

The magnitude of the frequency response is given by the following expression:

\[ |H(\Omega)| = \frac{1}{\sqrt{(1 + \alpha^2) - 2\alpha \cos(m\Omega)}} \]

Each of the graphs below shows a plot of the magnitude of the frequency response for this system, where \( \alpha \) is either 0.5 or 0.9, and \( m \) is either 5 or 9.

For each, indicate which values of \( \alpha \) and \( m \) correspond to the graph, and also label the minimum and maximum values of \( |H(\Omega)| \), as well as the \( \Omega \) values at which the minimum and maximum occur.
3. **Smiley**  [25 points]

The following image, $m[n_x, n_y]$, is 9 pixels wide and 9 pixels tall. Black pixels represent a value of 0 and white pixels represent a value of 1. The pixel in the center of the image is at the origin ($n_x = n_y = 0$). $n_y$ increases downward, and $n_x$ increases to the right.

The DTFT of $m[n_x, n_y]$ is given by the function $M(\Omega_x, \Omega_y)$.

Each of the following images measures 18 pixels wide and 18 pixels tall, and was synthesized from a DTFT that can be expressed in terms of the function $M$.

**Part 1.** Image 1

Write an expression for this image’s DTFT, $M_1(\Omega_x, \Omega_y)$, in terms of the function $M$.

$$M_1(\Omega_x, \Omega_y) =$$

**Part 2.** Image 2

Write an expression for this image’s DTFT, $M_2(\Omega_x, \Omega_y)$, in terms of the function $M$.

$$M_2(\Omega_x, \Omega_y) =$$
Part 3. Image 3

Write an expression for this image’s DTFT, $M_3(\Omega_x, \Omega_y)$, in terms of the function $M$.

$$M_3(\Omega_x, \Omega_y) =$$

Part 4. Image 4

Write an expression for this image’s DTFT, $M_4(\Omega_x, \Omega_y)$, in terms of the function $M$.

$$M_4(\Omega_x, \Omega_y) =$$

Part 5. Image 5

Write an expression for this image’s DTFT, $M_5(\Omega_x, \Omega_y)$, in terms of the function $M$.

$$M_5(\Omega_x, \Omega_y) =$$
4. Filtering  \[25\text{ points}\]

Part 1.

In lab 9b, Ben Bitdiddle wrote the following code to apply a “brick wall” low-pass filter to an image (under the assumption that the input image was square, with an odd number of pixels in each dimension). The code below is a working implementation with no bugs (line numbers are shown to the left).

```
01 | import numpy
02 | from image_utils import png_read, png_write
03 | from math import pi, cos, sin, e
04 |
05 | bird = png_read('bird.png')
06 | h, w = bird.shape
07 |
08 | assert h==w, 'Only works for square images'
09 | assert h % 2 == 1, 'Only works for images with odd dimensions'
10 |
11 | def make_lo_pass_dft(size, omega_cutoff):
12 |     kernel = numpy.zeros((size, size), dtype=float)
13 |     for r in range(-size//2, size//2+1):
14 |         for c in range(-size//2, size//2+1):
15 |             distance_from_center = (r**2 + c**2)**0.5
16 |             pixel_cutoff = omega_cutoff / (2 * pi) * h
17 |             if distance_from_center <= pixel_cutoff:
18 |                 kernel[r, c] = 1
19 |     return kernel
20 |
21 | output = ifft2(fft2(bird) * make_lo_pass_dft(h, pi/7))
```

When applying this filter to the image on the left (where 0 is black and 1 is white), Ben sees the image on the right.

Ben is pleased that the bird looks blurry, but he is upset by the “ringing” artifacts he notices after applying his low-pass filter to the image above.

In an effort to get rid of these artifacts, Ben decides to replace line 18 with the following code (leaving the other lines unchanged):

```
kernel[r, c] = cos((pi / 2) * distance_from_center / pixel_cutoff)
```
Will this change reduce the ringing artifacts in the output image produced by this code? Justify your answer with a brief explanation (2-4 sentences). If he is incorrect, what should he change to produce the desired result?

Each of the images below is normalized so that black represents the minimum value (not necessarily 0) and white represents the maximum value (not necessarily 1).

Which of the images below most closely matches the output Ben sees from running the code after making the change described above?

A

B

C

D

E

F

G

H

I

J

K

L
Part 2.

Ben’s friend, Lem E. Tweakit, realizes that he can create a high-passed version of the image by subtracting a low-passed version of the image from the original image. He decides to perform this operation in the spatial domain, using the following code:

```
01 | lo_pass_dft = make_lo_pass_dft(h, pi/7)
02 | lo_pass_space = ifft2(lo_pass_dft)
03 | hi_pass_space = 1 - lo_pass_space
04 | hi_pass_dft = fft2(hi_pass_space)
05 | output = ifft2(fft2(bird) * hi_pass_dft)
```

Will this change result in a high-passed version of the original image? Justify your answer with a brief explanation (2-4 sentences). If he is incorrect, what should he do differently to produce the desired result (still by working in the spatial domain)?

Which of the images from the previous page most closely matches the output Lem sees when running his code (assuming he uses Ben’s original `make_lo_pass_dft` function)?
Worksheet (intentionally blank)
Worksheet (intentionally blank)
Worksheet (intentionally blank)
Worksheet (intentionally blank)
Worksheet (intentionally blank)
Worksheet (intentionally blank)